On asymptotics of the structure on conditional configuration graphs with bounded number of links

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We consider configuration graphs where vertex degrees are independent identically distributed random variables. Configuration random graphs are being a good implementation of the social, telecommunication networks and Internet topology [1]. Numerous of real networks suggest that the distribution of degree ξ of each vertex can be specified by the relation

$$p_k = \mathbf{P}\{\xi = k\} = \frac{h(k)}{k^\tau}, \quad k = 1,2,..., \quad \tau > 1, \tag{1}$$

where $h(x) > 0$ is a slowly varying function. Let N be a number of vertices in the graph and random variables $\eta_1,...,\eta_N$ are equal to the degrees of vertices with the numbers $1,...,N$. We consider the subset of random graphs under the condition $\eta_1 + ... + \eta_N \le n$.

Denote by $\eta_{(N)}$ and μ_r the maximum vertex degree and the number of vertices with degree r respectively. Let $\tau > 3$. We introduce the following notations:

$$m = \mathbf{E}\,\xi = \sum_{k=1}^{\infty} \frac{h(k)}{k^{\tau-1}}, \qquad \sigma^2 = \mathbf{D}\,\xi = \sum_{k=1}^{\infty} \frac{h(k)}{k^{\tau-2}} - m^2.$$

The next theorems are proved.

Theorem 1. *Let $N, n \to \infty$ in such a way that*

$$\frac{n - Nm}{\sqrt{N}} \ge C > -\infty \tag{2}$$

and $r = r(N, n)$ are chosen such that

$$\frac{Nh(r)}{(\tau - 1)r^{\tau-1}} \to \gamma,$$

where γ is a positive constant. Then

$$\mathbf{P}\{\eta_{(N)} \le r\} \sim e^{-\gamma}.$$

Theorem 2. *Let $N, n \to \infty$, $Np_r(1 - p_r) \to \infty$ and*

$$\frac{n - Nm}{\sqrt{N}} \to \infty.$$

Then

$$\mathbf{P}\{\mu_r = k\} = \frac{1 + o(1)}{\sqrt{2\pi N p_r (1 - p_r)}} e^{-u^2/2}$$

uniformly in the integer k *such that* $u=(k-Np_r)/\sqrt{Np_r(1-p_r)}$ *lies in any fixed finite interval.*

Theorem 3. *Assume that* $N, n, r \to \infty$ *and condition* (2) *hold. Then*

$$\mathbf{P}\{\mu_r = k\} = \frac{(Np_r)^k}{k!} e^{-Np_r} (1+o(1))$$

uniformly in the integer k *such that* $(k-Np_r)/\sqrt{Np_r}$ *lies in any fixed finite interval.*

Proof strategy. Let $\xi_1,...,\xi_N$ be auxiliary independent identically distributed random variables with distribution (1). The technique for obtaining these theorems is based on the generalized allocation scheme suggested by V.F. Kolchin [2]. It is readily seen that for our subset of graphs

$$\mathbf{P}\{\eta_1 = k_1,...,\eta_N = k_N\} = \mathbf{P}\{\xi_1 = k_1,...,\xi_N = k_n \mid \xi_1 + ... + \xi_N \le n\}. \quad (3)$$

Let $\xi_i^{(r)}, \nu_i^{(r)}, i = 1,...,N,$ be two sets of independent random variables such that

$$\mathbf{P}\{\xi_i^{(r)} = k\} = \mathbf{P}\{\xi_i = k \mid \xi_i \le r\}, \quad \mathbf{P}\{\nu_i^{(r)} = k\} = \mathbf{P}\{\xi_i = k \mid \xi_i \ne r\}.$$

It is shown in [3, 4], that from (3) it is not hard to get:

$$\mathbf{P}\{\eta_{(N)} \le r\} = (1-\mathbf{P}\{\xi_1 > r\})^N \mathbf{P}\{\xi_1^{(r)} + ... + \xi_N^{(r)} \le n\} / \mathbf{P}\{\xi_1 + ... + \xi_N \le n\} \quad (4)$$

and

$$\mathbf{P}\{\mu_r = k\} = \binom{N}{k} p_r^k (1-p_r)^{N-k} \mathbf{P}\{\nu_1^{(r)} + ... + \nu_{N-k}^{(r)} \le n - kr\} / \mathbf{P}\{\xi_1 + ... + \xi_N \le n\}. \quad (5)$$

From (4) and (5) we see that to obtain the limit distributions of $\eta_{(N)}$ and μ_r it suffices to consider the asymptotic behavior of the sums of independent random variables, binomial $(1-\mathbf{P}\{\xi_1 > r\})^N$ and binomial probabilities. By this way we proved Theorems 1 – 3.

References

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