On the Properties of Estimates for the Gamma-Exponential Distribution Parameters

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A major role in applied problems solved by probability theory methods is played by the generalized gamma distribution $GG(v, q, \theta)$, or Kritsky–Menkel distribution (see Kritsky and Menkel [1]), which has a density

$$f(x) = \frac{|v|x^{vq-1}e^{-(x/\theta)^v}}{\theta^{vq}\Gamma(q)}, \quad v \neq 0, \quad q > 0, \quad \theta > 0, \quad x > 0.$$

The distributions of the gamma type are exponential distribution, Erlang distribution, gamma distribution, semi-normal distribution, Rayleigh distribution, Maxwell–Boltzmann distribution, χ-distribution, Nakagami m-distribution, Weibull–Gnedenko distribution, generalized Weibull distribution, pseudo Weibull distribution, inverse Rayleigh distribution, Frechet distribution etc.

Scale mixtures of generalized gamma distributions play an important part in many applied problems in various fields of knowledge, from queuing theory to emergency management methodology (see Kudryavtsev [2]). Their main probabilistic characteristics (density, distribution function, moments) were described in Kudryavtsev [3] for the case of different signs of the distribution form parameters. In Kudryavtsev [4] the following distribution, which is closely related to the generalized gamma distribution, was presented.

Definition. We say that the random variable ζ has the gamma-exponential distribution $GE(r, \nu, s, t, \delta)$ with the parameters of bent $0 \leq r < 1$, shape $\nu \neq 0$, concentration $s, t > 0$, and scale $\delta > 0$ if its density at $x > 0$ is

$$g_E(x) = \frac{|\nu|x^{t\nu-1}}{\delta^{t\nu}\Gamma(s)\Gamma(t)}\mathsf{Ge}_{r,\,tr+s}(-(x/\delta)^\nu), \tag{1}$$

where $E = (r, \nu, s, t, \delta)$, and $\mathsf{Ge}_{\alpha,\,\beta}(x)$ is the gamma-exponential function (see Kudryavtsev and Titova [5]):

$$\mathsf{Ge}_{\alpha,\,\beta}(x) = \sum_{k=0}^{\infty}\frac{x^k}{k!}\Gamma(\alpha k + \beta), \quad x \in \mathbb{R}, \quad 0 \leq \alpha < 1, \quad \beta > 0. \tag{2}$$

The function (2) generalizes to the case $\beta \neq 1$ the transformation introduced by Le Roy (see Le Roy [6]) to study generating functions of a special

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form. In addition, the function (2) can be considered (under some assumptions) as a special case of the Srivastava–Tomovski function (see Srivastava and Tomovski [7]), that generalizes the Mittag-Leffler function.

In Kudryavtsev [4] it was shown that the distribution (1) adequately describes Bayesian balance models (see Kudryavtsev [2]). This is primarily due to the fact that the distribution with the density (1) can be represented as a scaled mixture of two random variables with generalized gamma distributions.

Lemma. *1. Let the independent random variables λ and μ have the distributions $GG(v, q, \theta)$ and $GG(u, p, \alpha)$, $uv > 0$, respectively. Then the distribution of λ coincides with $GE(0, v, \cdot, q, \theta)$; the distribution of λ/μ for $|u| > |v|$ coincides with $GE(v/u, v, p, q, \theta/\alpha)$; the distribution of λ/μ for $|v| > |u|$ coincides with $GE(u/v, -u, q, p, \theta/\alpha)$.*

2. For $0 < r < 1$, the density $g_E(x)$, $E = (r, \nu, s, t, \delta)$, coincides with the density of the ratio of independent random variables with generalized gamma distributions $GG(\nu, t, \delta)$ and $GG(\nu/r, s, 1)$.

In practice, the researcher deals with observable quantities that reflect the evolution of the analyzed real process. In relation to these quantities, some model assumptions are made about the form of their distribution. The problem of estimating unknown parameters from real data also arises in the case of modeling a real process using the gamma-exponential distribution. Due to the representation of the density $g_E(x)$ in terms of a special gamma-exponential function (2), the maximum likelihood method seems to be too complicated. The same can be said about the direct method of moments. For this reason, in Kudryavtsev and Shestakov [8, 9] it was proposed to estimate the parameters of the gamma-exponential distribution using a modified method based on logarithmic moments.

The report describes a method for estimating the parameters of a scale mixture of generalized gamma distributions which is a special case of the gamma-exponential distribution. Analytical forms of parameter estimates are given, their statistical properties are discussed, and a software package that allows automatic estimation is demonstrated.

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