On the running maxima of some φ-subgaussian random double arrays

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The paper studies sufficient conditions on the tail distributions of $X_{k,n}$ that guarantee the existence of such a sequence $\{a_{m,j}, m \geq 1, j \geq 1\}$ that the random variables

$$Y_{m,j} = \max_{1 \leq k \leq m, 1 \leq n \leq j} X_{k,n} - a_{m,j}$$

converge to 0 almost surely as the number of random variables $X_{k,n}$ in the above maximum tends to infinity. This type of convergence is called the convergence of running maxima. First results of this type were obtained by Pickands [3], where the classical case of Gaussian random variables was considered. Later this result was generalized to wider classes of distributions. Running maxima of one-dimensional random sequences were considered by Giuliano [1] and the generalization to the subgaussian case was studied also. Giuliano, Ngamkham and Volodin [2] generalized the results of Giuliano [1] to the case of φ-subgaussian random variables. In this paper the convergence of the running maxima of centered double arrays with more general exponential types of the tail distributions of $X_{k,n}$ than in [2] is investigated. The integrability conditions on the subgaussian function φ will obviously change. Contrary to the majority of classical results on the limiting behaviour of the maxima of random variables, where the convergence in distribution was considered, we are interested in the almost surely convergence to zero.

Below some definitions that are used in the main results are introduced.

Definition 1. *A continuous function $\varphi(x), x \in \mathbb{R}$, is called an Orlicz N-function if*

a) it is even and convex,

b) $\varphi(0) = 0$,

c) $\varphi(x)$ is a monotone increasing function for $x > 0$,

d) $\lim\limits_{x \to 0} \frac{\varphi(x)}{x} = 0$ and $\lim\limits_{x \to +\infty} \frac{\varphi(x)}{x} = +\infty$.

Definition 2. *A function $\psi(x)$, $x \in \mathbb{R}$, given by $\psi(x) := \sup_{y \in \mathbb{R}} (xy - \varphi(y))$ is called the Young-Fenchel transform of $\varphi(x)$.*

Any Orlicz N-function $\varphi(x)$ can be represented in the integral form

$$\varphi(x) = \int_0^{|x|} p_\varphi(t)\, dt$$

where $p_\varphi(t)$, $t \geq 0$, is its density. The density $p_\varphi(\cdot)$ is non-decreasing and there exists a generalized inverse $q_\varphi(\cdot)$ defined by

$$q_\varphi(t) := \sup\{u \geq 0 : p_\varphi(u) \leq t\}.$$

Then,

$$\psi(x) = \int_0^{|x|} q_\varphi(t)\ dt.$$

As a consequence, the function $\psi(\cdot)$ is increasing, differentiable, and $\psi'(\cdot) = q_\varphi(\cdot)$.

Definition 3. *A random variable X is φ-subgaussian if $E(X) = 0$ and there exists a finite constant $a > 0$ such that $E\exp(tX) \leq \exp(\varphi(at))$ for all $t \in \mathbb{R}$. The φ-subgaussian norm $\tau_\varphi(X)$ is defined as*

$$\tau_\varphi(X) := \inf\{a > 0 : E\exp(tX) \leq \exp(\varphi(at)), t \in \mathbb{R}\}.$$

The class of subgaussian and φ-subgaussian random variables is a natural extension of the Gaussian class. Buldygin and Kozachenko [4] discussed subgaussianity and φ-subgaussianity in detail and provides numerous important examples and properties.

The main results of the paper (Theorems 1-3) are introduced below.
Let $\varphi(\cdot)$ be an Orlicz N-function, $p_\varphi(\cdot)$ be its density, the function $\psi(\cdot)$ is the Young-Fenchel transform of $\varphi(\cdot)$, and the function $q_\varphi(\cdot)$ be the generalized inverse of the density $p_\varphi(\cdot)$.

Let $\{X_{k,n}, k \geq 1, n \geq 1\}$ be a double array (2D random field defined on the integer grid $\mathbb{N} \times \mathbb{N}$) of zero-mean random variables. The next notations will be used to formulate the main results

$$\begin{aligned} Y_{m,j} : &= \max_{1 \leq k \leq m, 1 \leq n \leq j} X_{k,n} - a_{m,j}, \\ Z_{m,j} : &= X_{m,j} - a_{m,j}, \end{aligned}$$

where $a_{m,j}$ is an increasing function with respect to each of m and j variables, where $m, j \geq 1$.

Let

$$Y^+_{m,j} := \max(Y_{m,j}, 0) \ \text{ and } \ Y^-_{m,j} := \max(-Y_{m,j}, 0).$$

Theorem 1. *Let $\{X_{k,n}, k \geq 1, n \geq 1\}$ be a double array of φ-subgaussian random variables and $g(\cdot)$ be a non-decreasing function such that for all $k, n \geq 1$*

$$\tau_\varphi(X_{k,n}) \leq g(\ln(kn))$$

and

$$a_{m,j} = g(\ln(mj))\psi^{-1}(\ln(mj)).$$

Suppose that there exists an $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$

$$\int_0^\infty \psi(x) q_\varphi(x) \exp\left(-\frac{\varepsilon q_\varphi(x)}{g(\psi(x) + \ln(2))}\right) dx < +\infty.$$

Then $\lim_{m \vee j \to \infty} Y^+_{m,j} = 0$ a.s.

Theorem 2. *Let $\{X_{k,n}, k, n \geq 1\}$ be a double array of independent φ-subgaussian random variables and the array $\{a_{m,j}, m, j \geq 1\}$ and function $g(\cdot)$ are defined in Theorem 1. Let $\kappa(x)$ be a positive increasing differentiable function with the derivative $r(x) = \kappa'(x)$ non-decreasing for $x > 0$. Assume that there exists $C > 0$ such that for every $k, n \geq 1$ and all $x > 0$*

$$P\left(\frac{X_{k,n}}{g(\ln(kn))} < x\right) \leq \exp\left(-Ce^{-\kappa(x)}\right),$$

and

$$\psi(x) - \kappa\left(\frac{xg(x)}{g(0)}\right) \geq C_0(x)$$

for some function $C_0(\cdot)$. Suppose that there exists A, $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$

$$\int_A^{+\infty} \exp\left(-\frac{Cy}{2}\exp\left(-\kappa\left(\frac{g(\ln(y))}{g(0)}\psi^{-1}(\ln(y)) - \frac{\varepsilon}{g(\ln(y))}\right)\right)\right) < +\infty$$

and

$$\int_A^{+\infty} \psi(y)q_\varphi(y)\exp\left(\psi(y) - \frac{C}{2}\exp\left(C_0(y) + \frac{r\left(\frac{yg(\psi(y))}{g(0)} - \frac{\varepsilon}{g(\psi(y))}\right)}{g(\psi(y))}\right)\right) dy < +\infty.$$

Then $\lim_{m\vee j\to\infty} Y^-_{m,j} = 0$ *a.s.*

Theorem 3. *Assume that $\{X_{k,n}, k \geq 1, n \geq 1\}$ is a double array of independent φ-subgaussian random variables. If the assumptions of Theorems* 1 *and* 2 *are satisfied, then* $\lim_{m\vee j\to\infty} Y_{m,j} = 0$ *a.s.*

The conditions of the obtained results allow to consider a wide class of φ-subgaussian random fields and are weaker than even in the known results for the one-dimensional case.

References

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