On quantization dimensions of idempotent probability measures

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Quantization of a probability measure is an approximation of given measure by measures with finite supports. Within the framework of quantization theory, the concept of the quantization dimension $D(\mu)$ of the probability measure μ is defined and investigated. In Ivanov[1] it is shown that the quantization dimension can be defined according to the following general functorial scheme.

Let $\mathcal{F}$ be a seminormal metrizable functor in the category $Comp$ of compact Hausdorff spaces, (X, ρ) be a metric compactum, $\rho_{\mathcal{F}}$ be a functorial extension of the metric ρ onto $\mathcal{F}(X)$ and $\mathcal{F}_n(X)$ be the subspace of $\mathcal{F}(X)$, consisting of points ξ, such that support $\operatorname{supp}(\xi)$ contains at most n elements ($n \in \mathbb{N}$). The set $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n(X)$ is dense in $\mathcal{F}(X)$. For each $\xi \in \mathcal{F}(X)$ and $\varepsilon > 0$ put $N(\xi, \varepsilon) = \min\{n : \rho_{\mathcal{F}}(\xi, \mathcal{F}_n(X)) \leq \varepsilon\}$. For any point $\xi \notin \bigcup_{n\in\mathbb{N}} \mathcal{F}_n(X)$, the number $N(\xi, \varepsilon)$ increases indefinitely when $\varepsilon \to 0$. The rate of this increase is characterized by the value

$$\dim_{\mathcal{F}}(\xi) = \lim_{\varepsilon\to 0} \frac{\log N(\xi, \varepsilon)}{-\log \varepsilon},$$

which we call the dimension of finite approximation of the point ξ (if the specified limit does not exist, the upper and lower limits are considered, and we get the upper $\overline{\dim}_{\mathcal{F}}(\xi)$ or the lower $\underline{\dim}_{\mathcal{F}}(\xi)$ dimension of finite approximation).

If as $\mathcal{F}$ we take the exponent functor exp with the Hausdorff metric, then $\dim_{\mathcal{F}}(A)$ coincides with the box dimension $\dim_B A$ for any $A \in \exp(X)$. For the functor of probability measures P, the dimension of finite approximation $\dim_P(\mu)$ (determined by the Kantorovich – Rubinstein metric) coincides with the quantization dimension $D(\mu)$ of the measure $\mu \in P(X)$.

In idempotent mathematics, an analogue of a probability measure on a compactum X is a normed functional $\mu : C(X) \to \mathbb{R}$, linear with respect to idempotent arithmetic operations. As usual, $C(X)$ will denote the space of continuous functions on X. For a constant function on X with the value $c \in \mathbb{R}$, we use the notation c_X.

Definition 1. *A functional $\mu : C(X) \to \mathbb{R}$ is called an idempotent probability measure if for any $f, g \in C(X)$ and $c \in \mathbb{R}$*

1) $\mu(c_X) = c$;

2) $\mu(c_X + f) = c + \mu(f)$;

3) $\mu(\max\{f, g\}) = \max\{\mu(f), \mu(g)\}$.

As proved by M. Zarichnyi [2], the set $I(X)$ of idempotent probability measures on the compactum X endowed with the weak* topology is always a compact Hausdorff space. For any continuous mapping of compacta $f : X \to Y$, the mapping $I(f) : I(X) \to I(Y)$ is naturally defined. Thus, the construction I determines a covariant functor in the category *Comp*, which is normal in the sense of E.V. Shchepin. For any metric compact space (X, ρ) we define the metric ρ_I on $I(X)$ by the formula

$$\rho_I(\mu, \nu) = \sup\{\sum_{n=1}^{\infty} \frac{|\mu(nf) - \nu(nf)|}{n2^n} : f \in \mathrm{Lip}_1(X)\},$$

where $\mathrm{Lip}_1(X)$ is the set of real functions on X satisfying the Lipschitz condition with constant 1. The metric ρ_I determines the metrization of the functor I. (This metric is a modified version of the metric on $I(X)$, preposed in the paper L. Bazylevych, D. Repovš , M. Zarichnyi [3]. Our metric is more convenient for constructing a theory of quantization of idempotent probability measures.) Thus, for any idempotent probability measure $\mu \in I(X)$, the dimensions of finite approximation can be defined, which (by analogy with classical probability measures) we call the dimensions of quantization.

Definition 2. *For $\mu \in I(X)$ the quantization dimensions $\overline{D}_I(\mu)$ and $\underline{D}_I(\mu)$ (upper and lower) are:*

$$\overline{D}_I(\mu) = \overline{\dim}_I(\mu),\ \ \underline{D}_I(\mu) = \underline{\dim}_I(\mu).$$

We establish a number of properties of the quantization dimensions of idempotent probability measure $\mu \in I(X)$. In particular, the following inequalities are proved

$$\overline{D}_I(\mu) \leq \overline{\dim}_B(\mathrm{supp}(\mu));\ \ \underline{D}_I(\mu) \leq \underline{\dim}_B(\mathrm{supp}(\mu)).$$

This inequalities establish an upper bound on quantization dimensions of the measures $\mu \in I(X)$ with a given support.

We prove the following intermediate value theorem for the upper quantization dimension:

Theorem 1. *For any metric compactum (X, ρ) of dimension $\overline{\dim}_B X = a$ and any real number $b :\ 0 \leq b \leq a$ there exists a measure $\mu_b \in I(X)$ such, that $\overline{D}_I(\mu_b) = b$ and $\mathrm{supp}(\mu_b) = X$.*

For the lower quantization dimension a similar statement holds if the compactum X satisfies some additional conditions. For a metric compactum (X, ρ) we define the local lower box dimension $l\underline{\dim}_B X$ of X in the following way.

Let $x \in X$ and $B(x, \varepsilon)$ be the closed ε-ball of x. We put

$$l\underline{\dim}_B(X, x) = \inf\{\underline{\dim}_B B(x, \varepsilon) : \varepsilon > 0\},$$

$$l\underline{\dim}_B X = \sup\{l\underline{\dim}_B(X, x) : x \in X\}.$$

A subset $A \subset X$ is called *ε-separated* if $\rho(x, y) > \varepsilon$ for any two different points $x, y \in A$. We say that in a metric compactum (X, ρ) the *cardinality of ε-separated sets is locally bounded* if there exists $p \in \mathbb{N}$ such that any ε-separated subset of any ε-ball $B(x, \varepsilon)$ (where $\varepsilon > 0$, $x \in X$) contains at most p points.

Theorem 2. *If in a metric compactum (X, ρ) the cardinality of ε-separated sets is locally bounded and $l\underline{\dim}_B X = a > 0$, then for any nonnegative $b < a$ there exists a measure $\mu_b \in I(X)$ for which $\underline{D}_I(\mu_b) = b$ and* $\operatorname{supp}(\mu) = X$.

The last statement is based on the following intermediate value theorem for the lower box dimension:

Theorem 3. *Let in a metric compactum (X, ρ) the cardinality of ε-separated sets is locally bounded and $l\underline{\dim}_B X = a > 0$. Then for any nonnegative $b < a$ there exists a closed subset $Z_b \subset X$ such that $\underline{\dim}_B Z_b = b$.*

References

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