УДК 519.21

Generalization of one result of V.V. Senatov V.V. of characteristic functions of convolutions of probability

Sobolev V.N., Kondratenko A.E.

This paper gives new expansions of characteristic functions of convolution of symmetric probability distributions with an explicit estimate of the remainder.

Keywords: asymptotic expansion, characteristic function, accuracy of approximation, approximation exactness, estimate of the remainder, moments of the Chebyshev – Hermite, symmetric distribution of probabilities, symmetric random variables.

Let $\xi, \xi_1, \xi_2, ...$ be independent identically random variables with zero mean, unit variance. Throughout this paper the following notations will be used. The random variable ξ has symmetric around zero probability distribution P with the distribution function $F(x)$, the real characteristic function $f(t)$ and the finite moment $\mathrm{M}\xi^{m+2}$ of even order $m+2 \geqslant 2$. The normalized sum $(\xi_1 + ... + \xi_n)\, n^{-\frac{1}{2}}$ has the distribution function $F_n(x) = \mathsf{P}\left\{\frac{\xi_1+...+\xi_n}{\sqrt{n}} \leqslant x\right\}$ and the characteristic function $f^n\left(\frac{t}{\sqrt{n}}\right)$. The standardized normal distribution function is denoted by $\Phi(x)$ and $\varphi(x)$ is it's probability density function. By the same $\gamma(t)$ generally different real or complex quantities such that $|\gamma| \leqslant 1$.

The present paper is devoted to expansions of characteristic functions of convolution of distributions and estimates of its remainder.

We will use expansions containing the last known moment in the main part of the decomposition. They were proposed by H. Prawitz [2]. His result has been generalized by I.G. Shevtsova [6]. V.V. Senatov in [5, p. 189] has obtained two expansions of the real characteristic function. It is easily seen that such expansions can be written of any length

$$f(t) = \sum_{k=0}^{m} \alpha_k (it)^k + \lambda \alpha_{m+2} (it)^{m+2} + \overline{\lambda} \alpha_{m+2} (it)^{m+2} \gamma(t) \ ,$$

where $\alpha_k = \mathrm{M}\xi^k/k!$ and $\overline{\lambda} = \max\{\lambda;\, 1-\lambda\}$, $0 \leqslant \lambda \leqslant 1$.

When constructing expansions with an explicit estimate of the remainder in the CPT, it is convenient to use [7] the decomposition for the function $f(t)\, e^{\frac{t^2}{2}}$ in terms of the Chebyshev–Hermite moments $\theta_k = \theta_k(\mathrm{P}) = \dfrac{1}{k!}\int\limits_{-\infty}^{\infty} H_k(x) dF(x)$, where $H_k(x) = (-1)^k \varphi^{(k)}(x)/\varphi(x)$ is the Chebyshev–Hermite polynomials [1, p. 21]. At the same time, there are also the incomplete Chebyshev–Hermite moments $\theta_k^{(k-2)} = \sum\limits_{j=1}^{[k/2]} (-1)^j \alpha_{2j}(\varphi)\cdot\alpha_{k-2j}(\mathrm{P})$, the Chebyshev–Hermite moments with the parameter included in it (see [5]) $\theta_{m+2}^{(\lambda)} = \lambda\alpha_{m+2} + \theta_{m+2}^{(m)}$.

Lemma. *Let $f(t)$ is the characteristic function of symmetric distributions* P *with finit moments up to even $m+2 \geqslant 4$* inclusive. *Then*

$$f(t) = \left(\sum_{k=0}^{m} \theta_k (it)^k + \theta_{m+2}^{(\lambda)} (it)^{m+2} \right) e^{-\frac{t^2}{2}} +$$

$$+\gamma\overline{\lambda}\alpha_{m+2} |t|^{m+2} + \gamma \left\| \theta_{m+4}^{(m+2,\lambda)} \right\| |t|^{m+4} , \qquad (1)$$

where $\left\| \theta_{m+4}^{(m+2,\lambda)} \right\| = \lambda\alpha_{m+2}\alpha_2(\varphi) + \|\theta_m\|\, \alpha_4(\varphi)$.

From Lemma we can obtain the following Theorem.

Theorem. *Let $f(t)$ is the characteristic function of symmetric distributions* P *with finit moments up to even $m+2 \geqslant 4$* inclusive. *Then for $n \geqslant m+2$*

$$f^n\left(\frac{t}{\sqrt{n}}\right) = e^{-t^2/2} + e^{-t^2/2} \sum_{s=1}^{m/2} C_n^s \sum_{l=4s}^{m-4+4s} \Theta_{s,l} \left(\frac{it}{\sqrt{n}}\right)^l +$$

$$+\theta_{m+2}^{(\lambda)} \frac{(it)^{m+2}}{(\sqrt{n})^m} e^{-\frac{t^2}{2}} + R_{m+2} , \qquad (2)$$

where

$$|R_{m+2}| \leqslant \overline{\lambda} \frac{\alpha_{m+2}}{n^{m/2}} |t|^{m+2} \mu^{n-1} \left(\frac{t}{\sqrt{n}}\right) + O\left(\frac{|t|^{m+5}}{n^{m/2+1/2}}\right) , \qquad (3)$$

$$\Theta_{s,l} = \sum_{k_1+\ldots+k_s=l} \theta_{k_1} \ldots \theta_{k_s} . \qquad (4)$$

Summation in (4) is performed for all sets of natural numbers k_1,...,k_s that

$k_1 + ... + k_s = l$ and $k_j \geqslant 4$, $j = 1, ..., m-1$.

Moreover, there is the bound for $O\left(\frac{|t|^{m+5}}{n^{m/2+1/2}}\right)$ from (3). And last theorem can be used to build a new asymptotic expansions in the Central limit theorem with an explicit estimate of the remainder.

Список литературы

[1] Cramér H. “On the composition of elementary errors“ // Skandinavisk Aktuarietidskrift, 11 (1928), 13-74.

[2] Prawitz H. “Noch einige Ungleichungen fur charakteristische Funktionen“ // Scand. Actuar. J., 1 (1991), 49–73.

[3] Senatov V.V.. Normal Approximation: New Results, Methods and Problems. Utrecht:”VSP”, 1998, 362.

[4] Senatov V.V. and Sobolev V.N., “New forms of asymptotic expansions the central limit theorem”, Theory Probab. Appl., 57:1 (2013), 82–96.

[5] Senatov V.V., “On the real accuracy of approximation in the central limit theorem. II”, Siberian Adv. Math., 27:2 (2017), 133–152.

[6] Shevtsova I.G. “On the accuracy of the approximation of the complex exponent by the first terms of its Taylor expansion with applications“, J. Math. Anal. Appl., 418:1 (2014), 185–210.

[7] Sobolev V.N., “On expansion of the characteristic function of symmetric distributions”, Mathematical Physics and Computer Simulation, 21:1 (2018), 34–43.