Random Mappings with Constraints on the Component Sizes

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Let X be an arbitrary fixed finite set having n elements. By $\mathfrak{S}_n$ we denote a semigroup of all mappings from the set X into itself. Each mapping $\omega(\cdot) \in \mathfrak{S}_n$ corresponds to a graph $\Gamma(X, \sigma)$ whose vertices $x, y \in X$ are connected by an arc (x,y) if $y = \omega(x)$. As it's known, every graph $\Gamma(X, \omega)$ consists of connected components, each consisting of a single cycle and trees. Fix an arbitrary set D of natural numbers. By $\mathfrak{S}_n(D)$ denote a set of mappings from $\mathfrak{S}_n$ with component sizes belonging to the set D. Let a random mapping $\sigma = \sigma_n(D)$ be uniformly distributed on $\mathfrak{S}_n(D)$. This mapping is introduced in [10]. Random mappings with another restrictions were considered by a number of authors beginning from [9]. Brief reviews one can see in [3, 4, 5, 8].

Let as consider the following two classes of the sets D. We say that a set D belongs to the class F_1 iff $D = \bigcup_{i=1}^{M} D_i$, where $M \in N,\ D_i = \{m \in N :\ m = a_i k + b_i,\ k = 0, 1, 2, \dots\}$ and the integers $a_i > 1,\ 1 \le b_i \le a_i - 1,\ (a_i, b_i) = 1$ with $D_i \cap D_j = \emptyset,\ \forall i \ne j$. Also, we say that a set D belongs to the class F_2, iff $D = \{m \in N :\ m/k_i \notin N,\ i = 1, \dots, s\}$ for some $s \in N$ and $k_1, \dots, k_s \in N$ such that $k_i \ge 2,\ i = 1, \dots, s$ and $(k_i, k_j) = 1\ \forall i \ne j$.

It may be not obvious that there are a large number of mutually non-intersecting progressions $D_i,\ i = 1, \dots, M$ with different a_i, whose union belongs to the class F_1. Therefore, we give the following example.

$$D_1 = \{m \in N :\ m = 1 + 5k,\ k = 0, 1, 2, \dots\},$$

$$D_2 = \{m \in N :\ m = 2 + 15k,\ k = 0, 1, 2, \dots\},$$

$$D_3 = \{m \in N :\ m = 3 + 20k,\ k = 0, 1, 2, \dots\},$$

$$D_4 = \{m \in N :\ m = 4 + 25k,\ k = 0, 1, 2, \dots\}.$$

It is easy to see that the progressions D_1, D_2, D_3, D_4 belong to the class F_1 and are mutually disjoint.

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By $\varrho = \varrho(D)$, denote the density of the set D in the set of natural numbers:

$$\varrho(D) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi\{i \in D\} > 0.$$

Put

$$\alpha = 1 - \frac{\varrho}{2},\ l(n) = \sum_{i\in D(n)} \frac{1}{i},$$

where $D(n) = D \cap [1, n]$. Denote

$$f_\alpha(x) = (1-x)^{-\alpha} - 1, \quad u_\alpha(x) = (1-x)^{-\alpha}.$$

Let $\zeta_n = \zeta_n(D)$ be a total number of components in random mapping $\sigma = \sigma_n(D)$.

Theorem 1 *Suppose that $D \in F_1 \cup F_2$. Then, for some $\beta \in (0, 1/2]$*

$$\mathsf{E}\zeta_n = \frac{1}{2} l(n) + J(n) + O\left(\frac{\ln n}{n^\beta}\right)$$

as $n \to \infty$, where

$$J(n) = \frac{1}{2} \sum_{m\in D(n-1)} \frac{1}{m} f_\alpha\left(\frac{m}{n}\right) \to \frac{\varrho}{2} \int_0^1 \frac{f_\alpha(x)}{x}\, dx < \infty.$$

Theorem 2 *Let $D \in F_1 \cup F_2$. Then*

$$\mathsf{Var}(\zeta_n) = \mathsf{E}\zeta_n + \frac{1}{4} J_0(n) + \frac{1}{4} J_1(n) + O(n^{-\beta}\ln^2 n) + O(n^{-\nu}),$$

where

$$J_0 = \sum_{m,k\in D, m+k\ge n} \frac{1}{mk} u_\alpha\left(\frac{m}{n}\right) u_\alpha\left(\frac{k}{n}\right)$$

$$\to \varrho^2 \int_{(x,y)\in(0,1)^2: x+y\ge 1} (1-x)^{-\alpha}(1-y)^{-\alpha} \frac{dx\,dy}{xy} < \infty,$$

$$J_1 = \sum_{m,k\in D(n-1),\ m+k<n} \frac{1}{mk}\left(u_\alpha\left(\frac{m+k}{n}\right) - u_\alpha\left(\frac{m+k}{n} - \frac{mk}{n^2}\right)\right).$$

$$\to \varrho^2 \int_{x,y>0,\ x+y<1} \frac{1}{xy}(u_\alpha(x+y) - u_\alpha(x+y-xy))\, dx\, dy < \infty,$$

$$\nu = \frac{1-\alpha}{2(2-\alpha)}.$$

The proofs of Theorems 1 and 2 essentially use the results of the papers [7, 11].

Notation 1 *If the assumptions of Theorem 1 are satisfied, then*

$$l(n) = \varrho \ln n + c(D) + O\left(\frac{1}{n}\right),$$

as $n \to \infty$*. Here the constant* $c(D)$ *can be calculated explicitly [11].*

Corollary 1 *Under the assumptions of Theorem 1,*

$$\mathbf{D}\zeta_n \sim \mathbf{E}\zeta_n \sim \frac{\varrho}{2} \ln n.$$

as $n \to \infty$*.*

Let's say a few words on the previous results. It is shown in [2] that $\mathbf{E}\zeta_n = (\ln n)/2 + O(1)$ as $n \to \infty$ in the case $D = N$. Note also the paper [6]. One can deduce from here the correct in order estimate for the variance of the random variable ζ_n. The classes F_1 and F_2 were introduced in [1].

In this talk, we also give a survay on the asymptotic results obtained earlier for $\sigma_n(D)$.

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