On configuration model with bounded number of edgers

I. A. Cheplyukova[1]

[1]Karelian Research Centre RAS, Petrozavodsk, Russia, chia@krc.karelia.ru

We consider configuration random graphs with N vertices. The degrees of vertices are independent identically distributed random variables. The random variables $\xi_1,\ldots,\xi_N$ are equal to degrees of the vertices with the number $1,\ldots,N$. The degrees of the vertices are drawn independently from an arbitrary given distribution. Assume that we know only the limit behaviour of the tail of this distribution as $k\to\infty$:

$$\mathbf{P}\{\xi_i=k\}=\frac{d}{k^g(\ln k)^h},$$

where $i=1,\ldots,N, \quad d>0, g>1, h\geqslant 0$. These graphs were first studied in [1].

We consider a subset of this graphs under the condition that the sum of vertex degrees at most n. Denote by $\eta_1,\ldots,\eta_N$ the random variables equal to the degrees of vertices in such a conditional random graph. It is evident that these random variables are dependent, and for natural $k_1,\ldots,k_N$ such that $k_1+\ldots+k_N\leq n$

$$\mathbf{P}\{\eta_1=k_1,\ldots,\eta_N=k_N\}=\mathbf{P}\{\xi_1=k_1,\ldots,\xi_N=k_N|\xi_1+\ldots+\xi_N\leq n\}. \quad (1)$$

The equation (1) means that for the random variables $\xi_1,\ldots,\xi_N$ and $\eta_1,\ldots,\eta_N$ the analogue of the generalized allocation scheme is valid (see [2]). The generalized allocation scheme itself was investigated and suggested by V.F.Kolchin [3]. It is shown in [2,4], that for the maximum vertex degree $\eta_{(N)}$ and μ_r equal to the number of vertices of degree r is true

$$\mathbf{P}\{\mu_r=k\}=\binom{N}{k}p_r^k(1-p_r)^{N-k}\frac{\mathbf{P}\{\zeta_{N-k}^{(r)}\leq n-kr\}}{\mathbf{P}\{\zeta_N\leq n\}} \quad (2)$$

and

$$\mathbf{P}\{\eta_{(N)}\leqslant r\}=(1-P_r)^N\frac{\mathbf{P}\{\tilde{\zeta}_N^{(r)}\leq n\}}{\mathbf{P}\{\zeta_N\leq n\}}, \quad (3)$$

where $\zeta_N=\xi_1+\ldots+\xi_N, \zeta_{N-k}^{(r)}=\xi_1^{(r)}+\ldots+\xi_{N-k}^{(r)}, \tilde{\zeta}_N^{(r)}=\tilde{\xi}_1^{(r)}+\ldots+\tilde{\xi}_N^{(r)}, P_r=\mathbf{P}\{\xi_1>r\}$ and two sets of independent identically distributed random variables $\xi_1^{(r)},\ldots,\xi_N^{(r)}$ and $\tilde{\xi}_1^{(r)},\ldots,\tilde{\xi}_N^{(r)}$ such that

$$\mathbf{P}\{\xi_i^{(r)}=j\}=\mathbf{P}\{\xi_1=j|\xi_1\neq r\},\quad j=1,2,\ldots,\quad i=1,\ldots,N,$$

$$\mathbf{P}\{\tilde{\xi}_i^{(r)}=j\}=\mathbf{P}\{\xi_1=j|\xi_1\leqslant r\},\quad j=1,2,\ldots,\quad i=1,\ldots,N.$$

Based on equations (2) and (3) we obtained the limit distributions of $\eta_{(N)}$ and μ_r in these conditional configuration graphs under various types of behaviour N and n tending to infinity. Below we will give one of the theorems. Let

$$B_N = \begin{cases} (N(g-1)^h/\ln^h N)^{1/(g-1)}, & 1 < g < 3; \\ \sqrt{N \ln^{1-h} N}, & g = 3, h < 1; \\ \sqrt{N \ln \ln N}, & g = 3, h = 1; \\ \sigma\sqrt{N}, & g > 3 \quad \text{or} \quad g = 3, h > 1, \end{cases}$$

$$m = \mathbf{E}\xi_1, \qquad \sigma^2 = \mathbf{D}\xi_1.$$

Theorem 1. *Let $N, n \to \infty$ and one of the following conditions hold:*

1. $1 < g < 2, n/B_N \geqslant C > 0$;
2. $g = 3, 0 \leqslant h \leqslant 1, (n - Nm)/B_N \geqslant -C > -\infty$;
3. $g = 3, h > 1, (n - Nm)/B_N \to \infty$
   $g = 3, h > 1, r = m, (n - Nm)/B_N \geqslant -C > -\infty$;
4. $g > 3, (n - Nm)/B_N \to \infty$
   $g > 3, r = m, (n - Nm)/B_N \geqslant -C > -\infty$.

Then for any fixed natural r

$$\mathbf{P}\{\mu_r = k\} = \frac{e^{-u_r^2/2}}{\sqrt{2\pi N p_r(1-p_r)}}(1 + o(1)))$$

uniformly in the integers $k \geqslant 0$ such that $u_r = (k - Np_r)/\sqrt{Np_r(1-p_r)}$ lies in any fixed finite interval.

References

1. Yu. L. Pavlov, Conditional configuration graphs with random parameter of the power-law degree distribution, *Sbornik: Mathematics* **209**:2 (2018) 258–275.

2. A. N. Chuprunov, I. Fazecas, An analogue of the generalized allocation scheme: limit theorems for the number of cells containing a given number of particles, *Discrete Mathematics and Applications* **22**:1 (2012) 101–122.

3. V. F. Kolchin, *Random Graphs*, Cambridge Univ. Press, Cambridge, 2010.

4. A. N. Chuprunov, I. Fazecas, An analogue of the generalized allocation scheme: limit theorems for the maximum cell load, *Discrete Mathematics and Applications* **22**:3 (2012) 307–314.