Investigation of the existence of a solution for stochastic differential equations in the class L_p, for $1 < p < 2$

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This work is devoted to the study of the existence and uniqueness of the solution of a stochastic differential equation in L_p, for $1 < p < 2$. The classical theorem for the space L_2 is widely known if

$$\left|b\left(t,x\right)-b\left(t,y,\right)\right|+\left|\sigma\left(t,x\right)-\sigma\left(t,y\right)\right|\leqslant L\left|x-y\right|$$

and $b(t,x)^2+\sigma(t,x)^2\leqslant c\left(1+x^2\right), x\in\mathbb{R}, t\in[0,T]$, then the solution to the SDE

$$dX_t=b\left(t,X_t\right)dt+\sigma\left(t,X_t\right)dW_t,\qquad 0\leqslant t\leqslant T, X_0=Z$$

exists and is unique in the space L_2 [1]. Let us investigate the existence of a solution in the space L_p. Let $X_t^0=Z, t\in[0,T]$ and for $n\geqslant 1$

$$X_t^{(n)}=Z+\int\limits_0^t b\left(s,X_s^{(n-1)}\right)ds+\int\limits_0^t \sigma\left(s,X_s^{(n-1)}\right)dW\left(s\right).$$

1) Proof of progressive measurability for drift functions $b\left(s,X_s\right)$, diffusion $\sigma\left(s,X_s\right)$ and $X_t^{(n)}$ is carried out similarly [1]. Let us prove that they lie in L_p. Let us find $\mathbb{E}\left(X_t^{(n)}\right)^p$. We successively apply Jensen's inequality and Jensen's integral inequality.

$$\mathbb{E}\left(X_t^{(n)}\right)^p\leqslant 3^{p-1}\left(\mathbb{E}Z^p+\mathbb{E}\left(\int\limits_0^t b\left(s,X_s^{(n-1)}\right)ds\right)^p+\mathbb{E}\left(\int\limits_0^t \sigma\left(s,X_s^{(n-1)}\right)dW_s\right)^p\right)$$

$$\leqslant 3^{p-1}\left(\mathbb{E}Z^p+T^{p-1}\int\limits_0^t \mathbb{E}\left(b\left(s,X_s^{(n-1)}\right)\right)^p ds+\mathbb{E}\left(\int\limits_0^t \sigma\left(s,X_s^{(n-1)}\right)dW_s\right)^p\right) \quad (1)$$

Applying to the third term sequentially the martingale inequality from [2] and Jensen's inequality for the expectation, taking into account that for $p/2$, and for $1 < p < 2$ it is a concave function

$$\mathbb{E}\left(\int_0^t \sigma\left(s, X_s^{(n-1)}\right) dW_s\right)^p \leqslant B_p \left(\int_0^t \mathbb{E}\left|\sigma\left(s, X_s^{(n-1)}\right)\right|^2 ds\right)^{p/2}$$

As you can see, if the diffusion and drift functions satisfy the condition $b(t,x)^2 + \sigma(t,x)^2 \leqslant c\,(1 + x^p)\,, x \in \mathbb{R}, t \in [0,T]\,, 1 \leqslant p < 2$,then the second and third integrals in (1) are bounded, and for an arbitrary $1 \leqslant p < 2$ we obtain that $X_t^{(n)}$ belong to the space L_p

2)Now let's start investigating the convergence of this sequence.

$$\mathbb{E}\left|X_t^{(1)} - X_t^{(0)}\right|^p = \mathbb{E}\left|\int_0^t b\,(s,Z)\,ds + \int_0^t \sigma\,(s,Z)\,dW_s\right|^p \leqslant$$

$$\leqslant 2^{p-1}\mathbb{E}\left|\int_0^t b\,(s,Z)\,ds\right|^p + 2^{p-1}\mathbb{E}\left|\int_0^t \sigma\,(s,Z)\,dW_s\right|^p \leqslant$$

$$\leqslant 2^{p-1}T^{p-1}\int_0^t \mathbb{E}\,(b\,(s,Z))^p\,ds + 2^{p-1}B_p\left(\int_0^t \mathbb{E}\left|\sigma\,(s,Z)\right|^2 ds\right)^{p/2} \leqslant$$

$$\leqslant M_1 t + M_2 t^{p/2} \leqslant M_1 t + M_2 t \leqslant Mt, \text{assume } M_1, M_2, M - \text{some constants}$$

For the n-th approximation we have:

$$\mathbb{E}\left|X_t^{(n+1)} - X_t^{(n)}\right|^p \leqslant$$

$$\leqslant \mathbb{E}\left|\int_0^t \left(b\left(s, X_s^{(n)}\right) - b\left(s, X_s^{(n-1)}\right)\right) ds + \int_0^t \left(\sigma\left(s, X_s^{(n)}\right) - \sigma\left(s, X_s^{(n-1)}\right)\right) dW_s\right|^p$$

$$\leqslant 2^{p-1}\mathbb{E}\left(\int_0^t L\left|X_s^{(n)} - X_s^{(n-1)}\right| ds\right)^p + 2^{p-1}\mathbb{E}\left|\int_0^t \left(L\left|X_s^{(n)} - X_s^{(n-1)}\right|\right) dW_s\right|^p$$

Now it is necessary to estimate the last integral from above

$$\mathbb{E}\left|\int_0^t \left(L\left|X_s^{(n)} - X_s^{(n-1)}\right|\right) dW_s\right|^p \leqslant C_p \mathbb{E}\int_0^t \left|L\left|X_s^{(n)} - X_s^{(n-1)}\right|\right|^p ds$$

The correctness of the inverse estimate for this martingale inequality was proved in [3] for $1 < p < 2$.

In order to prove that there are functions for which our estimate is applicable, a software package for the numerical simulation of stochastic processes was written and it was shown that such functions exist. The simulation was carried out using three different methods, direct stochastic Monte Carlo simulation for a random process, as well as calculating the change in density over time using finite difference and finite element methods.

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References

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