Ergodicity of birth and death processes on $\mathbb{Z}$ and double-ended queues

Y. A. Satin[2] ***A. I. Zeifman***[1], ***I. A. Kovalev***[2], ***R. V. Razumchik***[3]

[1] Vologda State University; Federal Research Center "Computer Science and Control" of RAS; Vologda Research Center of RAS, Moscow Center for Fundamental and Applied Mathematics, Moscow State University, Russia, a_zeifman@mail.ru
[2] Vologda State University, Vologda, Russia, yacovi@mail.ru, kovalev.iv96@yandex.ru
[3] Institute of Informatics Problems of the FRC CSC RAS, Moscow Center for Fundamental and Applied Mathematics, Russian Federation, rrazumchik@ipiran.ru

In this short note we revisit the problem of the computation of the limiting distribution of the number of unpaired customers in one specific class of double-ended queueing systems (QS) — Markov double-ended queues with times varying rates. Despite the fact it is now more than 50 years since such QS were firstly introduced, they keep attracting attention from the operation research community, as can be seen from Girno et al [1] and Wang at al [3]. The double-ended QS being considered here is one the typical types. There are two queues (say, I and II) of infinite capacity, running in parallel, each with a dedicated arrival flow. Customer from both queues are served one-by-one (say in FIFO order) by the single server under the following restrictions: (i) the service takes negligible amount of time and (ii) in order to perform this service the server must take one customer from each queue. Whenever one (or both) of the queues is empty, the server remains idle. Although it is unimportant for the questions discussed further, we mention that the described QS is not work-conserving. A plenty of results are available out there for the case when the underlying processes are exponential or phase-type with time-independent rates. But the Markovian case, when all the rates are non-random functions of time, is less studied (for a short review one can refer to Girno et al [1] and Wang at al [3]). Here we complement the available research results by showing that the well-established theory of stability of differential equations systems allows one to obtain some useful probability bounds for Markovian double-ended QS with all the rates varying with time.

Let $X(t)$ be the number of unpaired customers in the queue I. Then the state space of $X(t)$ is $\{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$. If $X(t) > 0$ ($X(t) < 0$) the number of unpaired customers is $|X(t)|$ and all of them reside in the the queue I (queue II). If $X(t) = 0$ the system is empty. Assume the arrival process to both queues is Poisson, but the rate to the queue I is $\lambda_{X(t)}(t)$ and to the queue II is $\mu_{X(t)}(t)$ i.e. the rates are non-random functions of time and may depend on the queue size. Omitting the argument t, the transition diagram of $X(t)$, which is the inhomogeneous birth and death process on $\mathbb{Z}$, is shown in the Fig. 1.

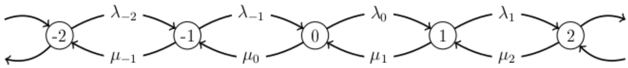

Figure 1: The transition diagram for $X(t)$

Let $p_k(t)$ denote the probability that $X(t)$ is in the state k at time t. The probabilistic dynamics of the $X(t)$ is given by the forward Kolmogorov system of differential equations:

$$\frac{d}{dt}p_k(t) = \lambda_{k-1}(t)p_{k-1}(t) - (\lambda_k(t) + \mu_k(t))\, p_k(t) + \\ + \mu_{k+1}(t)p_{k+1}(t), \; k = \ldots, -1, 0, 1 \ldots \quad (1)$$

It turns out that using this system alone (under some assumptions of $\lambda_k(t)$ and $\mu_k(t)$) it is possible to find the upper bounds for the rate of convergence of the probability distribution $\{p_k(t), k \in \mathbb{Z}\}$ to the limiting (assuming that it exists), and also provide two-sided truncations of the countable state space, which allows one to perform the numerical solution of (1) with the required accuracy.

The first step is to transform the Kolmogorov differential equations in a special way, removing the state $\{0\}$ from the consideration. The technique used for this one can see at Satin at al [2] and Zeifman et al [4]. After such a transformation, the linear system of differential equations (1) takes the form $\frac{d}{dt}\vec{x}(t) = B^*(t)\vec{x}(t)$, where $\vec{x}(t)$ is no longer the probabilistic vector and $B^*(t)$ is some matrix with the elements of arbitrary signs. Then one applies to this new system the method of the logarithmic norm (see Zeifman et al [4]), from which the ergodicity bounds for $\{p_k(t), k \in \mathbb{Z}\}$ follow. Among the other results, the described approach leads to the explicit estimates for the (limiting) probabilities of $X(t)$ to be in some subsets of the state space; for example,

$$\lim_{t\to\infty} \sup \mathsf{P}\{|X(t)| < N\} \le c_n,$$

where the constants c_n can be explicitly computed (under certain regularity conditions on $\lambda_k(t)$ and $\mu_k(t)$). It is important to notice, that the logarithmic norm method works if the matrix $B^*(t)$ is essentially non-negative. But it turns out that for the described birth and death processes on $\mathbb{Z}$ it is always the case.

Acknowledgement. This research was supported by Russian Science Foundation under grant 19-11-00020.

References

1. V. Giorno, A. Nobile. On a class of birth-death processes with time-varying intensity functions, *Applied Mathematics and Computation* (2020), 379, 125255.

2. Y. Satin, A. Korotysheva, K. Kiseleva, G. Shilova,E. Fokicheva ,A. Zeifman, V. Korolev. Two-Sided Truncations Of In- homogeneous Birth-Death Processes, *ECMS 2016 Proceedings edited by: Thorsten Claus, Frank Herrmann, Michael Manitz, Oliver Rose European Council for Modeling and Simulation* (2016), doi:10.7148/20160663.

3. Z. Wang, Ch. Yang, L. Liu, Q. Zhao. Equilibrium and Socially optimal of a double-sided queueing system with two-mass, (2021), https://arxiv.org/abs/2101.12043

4. A. Zeifman, Y. Satin, A. Kryukova, R. Razumchik, K. Kiseleva, G. Shilova. On the Three Methods for Bounding the Rate of Convergence for some Continuous-time Markov Chains, *International Journal of Applied Mathematics and Computer Science* (2020), **30**, 251–266.