An application of multi-server loss system with general service times

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In the present work a generalized multi-server loss model is considered. First we presented a blocking model proposed by W.Whitt, in which customer requires servers on several stations simultaneously. Then we consider another interpretation of this model with application to bandwidth sharing. Specifically we propose to use this model to obtain useful bounds of the blocking probabilities and the stationary number of users in the system. We note that these results are important since they hold for general service times. Also we compare our results with known results for exponential service times by means of some numerical examples.

First we consider a multi-class loss system studied in the paper [1]. There are n facilities (stations) in this system and facility i has S_i servers. Let we have c classes of customers, and a class-j customer requires exactly one server at each station of a subset of servers A_j. Also we assume that class-j customers follow Poisson input process with rate λ_j and have service times with general distribution and finite mean $1/\mu_j$. We denote by N_j the stationary number of class-j customers in the system and N_j^∞ the stationary number of class-j customers in a similar system, given that each station has *infinite number of servers*. The stationary distribution of $(N_1, \dots, N_c)$ is given by:

$$\mathsf{P}(N_j = k_j, 1 \le j \le c) = \mathsf{P}\Big(N_j^\infty = k_j, 1 \le j \le c \,\Big|\, \sum_{j \in C_i} N_j^\infty \le s_i, 1 \le i \le n\Big) = \\ = \frac{\mathsf{P}(N_j^\infty = k_j, 1 \le j \le c)}{\mathsf{P}(\sum_{j \in C_i} N_j^\infty \le s_i, 1 \le i \le n)}. \quad (1)$$

The stationary distribution of vector $(N_1^\infty, \dots, N_c^\infty)$ is known and given by:

$$\mathsf{P}(N_j^\infty = k_j, 1 \le j \le c) = \prod_{j=1}^{c} \mathsf{P}(N_j^\infty = k_j) = \prod_{j=1}^{c} \frac{\rho_j^{k_j}}{k_j!} e^{-\rho_j}, \quad (2)$$

where $\rho_j = \lambda_j/\mu_j$ is the traffic intensity.

Now we present another interpretation of the model described above. Let we have an Internet broadband connection and a class-k user needs b_k units of the *basic digital channels*, $k = 1, \dots, K$. We assume that class-k users follow Poisson process with rate λ_k and service times are exponentially distributed with mean $1/\mu_k$. This model is considered in works [2, 3], where the stationary distribution of the number of class-k users in the system is obtained. We denote $\mathbf{n} = (n_1, \ldots, n_K)$ the vector state of the system, where n_i is the number of class-i users in the system, and we denote by S the state space. Then the stationary distribution is given by

$$p(\mathbf{n}) := P(n_1, \ldots, n_K) = \frac{1}{G} \prod_{k=1}^{K} \frac{\rho_k^{n_k}}{n_k!}, \quad \mathbf{n} \in S, \tag{3}$$

$$G = \sum_{\mathbf{n} \in S} \prod_{k=1}^{K} \frac{\rho_k^{n_k}}{n_k!},$$

and $\rho_k = \lambda_k/\mu_k$ is the offered load.

Let us consider numerical example of this model, in which there are $n = 3$ servers $b_1 = 2$ and $b_2 = 1$. We obtain probabilities for exponential case and calculate the blocking probabilities using formula (3):

$$G = (1 + \rho_2 + \frac{\rho_2^2}{2} + \frac{\rho_2^3}{6}) + \rho_1(1 + \rho_2),$$
$$\pi(1) = \frac{1}{G}(\rho_1(1 + \rho_2) + \frac{\rho_2^2}{2} + \frac{\rho_2^3}{6}), \quad \pi(2) = \frac{1}{G}(\rho_1\rho_2 + \frac{\rho_2^3}{6}),$$

where $\pi(k)$ is the stationary probability that user of class-k will be lost.

Now we present an application of using Whitt formula (1) for similar model. First we assume that class-k consist of $\binom{n}{b_k}$ subclasses of customers, and we denote by J_k the set of these subclasses. For each subclass $j \in J_k$ we denote by $A_j = \{j_1, j_2, \ldots, j_{b_K}\}$ the set of the required stations, that is, the subset A_j is one of $\binom{n}{b_k}$ combinations of the stations numbers. Now we consider these subclasses as the classes of customers in the Whitt blocking model. Then, we split the arrival Poisson process to several Poisson processes with equal arrival rates. In the end, we use Whitt formula (1) to obtain the stationary distribution, and then return to original classes by summing up the probabilities of the corresponding subclasses.

Let us consider the same example as presented above to compare these systems. The subsets A_j are given by:

$$\begin{aligned} A_1 &= \{1, 2\}, \quad A_4 = \{1\}, \\ A_2 &= \{1, 3\}, \quad A_5 = \{2\}, \\ A_3 &= \{2, 3\}, \quad A_6 = \{3\}, \end{aligned} \tag{4}$$

Then we obtain the stationary probabilities using formula (1) and blocking probabilities:

$$L = \rho_1(1 + \frac{\rho_2}{3}) + (1 + \frac{\rho_2}{3})^3,$$
$$b(1) = \frac{1}{L}(\rho_1(1 + \frac{\rho_2}{3}) + \frac{\rho_2^2}{3}), \quad b(2) = \frac{1}{L}\frac{\rho_2}{3}(\rho_1 + \frac{\rho_2^2}{9}),$$

where $b(k)$ is the stationary probability that user of class-k will be lost.

The numerical examples show that these systems are slightly different because, in the Whitt model, there are some additional limitations for arriving customer to occupy a server. Therefore, the probability that the number of users in the original system is 0 is less than the probability obtained using Whitt formula, while all other probabilities are larger. Also we suppose that this analysis can potentially give useful bounds of the blocking probabilities for the system with general service times distribution. On the other hand, we note that this system is interesting by itself for applications where customer requires strictly defined numbers of servers.

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