Stability of a Multiclass Queueing System with Batch Markovian Arrival Process

Alexander S. Rumyantsev[1]***, Srinivas R. Chakravarthy***[2]

[1]Institute of Applied Mathematical Research, Karelian Research Centre of RAS, Petrozavodsk, Russia, ar0@krc.karelia.ru
[2]Departments of Industrial and Manufacturing Engineering & Mathematics, Kettering University, Flint, MI 48504, USA, schakrav@kettering.edu

In this talk, we address the problem of stability analysis of a structured continuous-time Markov chain (CTMC) known as the level-independent M/G/1-type process. It is a two-dimensional process

$$\{X(t), Y(t)\}_{t\geqslant 0}, \tag{1}$$

such that the *level*, $X(t)$, is a nonnegative integer-valued variable decreased by at most one at each transition, whereas the *phase*, $Y(t)$, may take any value from the (finite) *phase state space*, $\mathcal{Y}$, upon transitions.

The two-dimensional structure of the process allows to enumerate the states lexicographically into pairs (x, y), where x is the level value, and y is the phase value (and we use this multi-dimensional lexicographically ordered index to enumerate the state space of the CTMC). Using this enumeration, the infinitesimal generator matrix $\boldsymbol{Q}$ can be represented in the block semi upper-triangular form

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{A}^{0,0} & \boldsymbol{A}^{0,1} & \boldsymbol{A}^{0,2} & \boldsymbol{A}^{0,3} & \dots \\ \boldsymbol{A}^{1,0} & \boldsymbol{A}^{1,1} & \boldsymbol{A}^{(1)} & \boldsymbol{A}^{(-1)} & \dots \\ \mathbb{O} & \boldsymbol{A}^{(-1)} & \boldsymbol{A}^{(0)} & \boldsymbol{A}^{(1)} & \dots \\ \mathbb{O} & \mathbb{O} & \boldsymbol{A}^{(-1)} & \boldsymbol{A}^{(0)} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2}$$

where the element $\boldsymbol{Q}_{(x_0,y_0),(x_1,y_1)}$ of the matrix is the *transition rate* of the process $\{X(t), Y(t)\}_{t\geqslant 0}$ from the state (x_0, y_0) to the state (x_1, y_1), $\boldsymbol{A}^{0,i}$ are (not necessarily square) matrices consisting of transition rates from the (boundary) level 0, whereas $\boldsymbol{A}^{1,0}$ consists of transition rates to level 0. The matrices below the main diagonal, $\mathbb{O}$, are zero matrices. The matrices $\boldsymbol{A}^{(i)}$ are square matrices consisting of transition rates from level x to $x+i$, $i = -1, 0, 1, \dots$. It is worth noting that only the diagonal elements of $\boldsymbol{Q}$ are negative, and $\boldsymbol{A} = \sum_{i=-1}^{\infty} \boldsymbol{A}^{(i)}$ is a generator matrix of a finite state CTMC. Commonly in the queueing applications, the level corresponds to the number of customers in the system or in the queue. Then transitions from level x to the level $x-1$ and the level $x+i$ correspond to a departure and an arrival of a batch of size $i = 1, 2, \dots,$ respectively.

Being a special case of a CTMC, the irreducible M/G/1-type process (1) has a specific form of the Foster negative drift criterion: if $\sum_{k=1}^{\infty} k\boldsymbol{A}^{0,k}\mathbf{1} < \infty$, the M/G/1-type process is stable iff

$$\boldsymbol{\alpha}\sum_{i=1}^{\infty} i\boldsymbol{A}^{(i)}\mathbf{1} < \boldsymbol{\alpha}\boldsymbol{A}^{(-1)}\mathbf{1}, \tag{3}$$

where $\boldsymbol{\alpha} \geqslant \mathbf{0}$ is the unique *stochastic* vector, solution of the linear system

$$\boldsymbol{\alpha}\boldsymbol{A} = \mathbf{0}, \tag{4}$$

see e.g. He [1]. It is worth noting that (3) does not involve the matrices $\boldsymbol{A}^{i,j}, i = 0, 1, j = 0, 1, \ldots$ containing transition rates to boundary states. Moreover, it follows from (4) that $\boldsymbol{\alpha}$ is the steady-state probability vector of the finite-state CTMC governing the phase transitions *at high levels*.

A special case of the M/G/1-type process is the so-called quasi-birth-death (QBD) process having the matrices $\boldsymbol{A}^{(i)} \equiv \mathbb{O}, i \geq 2$. For such a process, in a recent paper Rumyantsev [2] the following result was established: if $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(0)}$ are diagonal, then the stability criterion can be shown to have the following form:

$$\sum_{y\in\mathcal{Y}} \boldsymbol{\pi}_y \rho_y < 1, \tag{5}$$

where

$$\rho_y = \frac{A^{(1)}_{y,y}}{\boldsymbol{d}_y}, \quad y \in \mathcal{Y},$$

and $\boldsymbol{d} = \boldsymbol{A}^{(-1)}\mathbf{1} \geq \mathbf{0}$ is the summary per-phase output rate related to level decrease of the process. For a multiclass system, the value $\boldsymbol{\pi}_y$ can be shown to have a product form.

In this talk we extend the result towards the process of M/G/1-type with infinitesimal generator matrix of the form (2) that models the multiclass multi-server system with the so-called batch Markovian arrival process as the input.

Acknowledgements The work of AR is partially supported by RFBR, projects 19-07-00303, 19-57-45022.

References

1. Q.-M. He, *Fundamentals of Matrix-Analytic Methods*, Springer New York, 2014.

2. A. Rumyantsev, Stability of Multiclass Multiserver Models with Automata-type Phase Transitions, *Proceedings of the Second International Workshop on Stochastic Modeling and Applied Research of Technology (SMARTY 2020)*. CEUR-WS, **2792** (2020) 213–225.