Symbolic-numeric scheme for calculating the moment characteristics of the state vector of stochastic differential-difference system

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A lot of technical, natural and economic systems has the property of aftereffect, which means that future states depend not only on the present but also on the past. It has long been established that the presence of aftereffect must be taken into account in models of mechanical, physical, chemical, biological and other systems, when solving problems of control theory, medicine, atomic energy, information, etc.

Probabilistic mathematical models of the phenomena are based on stochastic ordinary differential-difference equations (SODDE) that are generalizations of both deterministic equations with constant delays and stochastic ordinary differential equations Rubanik [1], Mao [2].

Currently, a significant number of research areas are associated with calculations of statistical characteristics of stochastic processes that are solutions of SDDE (see, for example, Buckwar [3], Elbeyli, Sun and Ünal [4], Poloskov [5]). But getting such solutions is quite difficult. Therefore, an urgent task is to develop effective both direct, i.e., obtaining trajectories of movements, and indirect, i.e., finding probabilistic characteristics, approximate numerical algorithms for the analysis of SODDE systems.

The paper is devoted to presentation of symbolic-numeric scheme for calculating the moment functions of a random vector process. Suppose this process is governed by the system of SODDE in the following form:

$$
\begin{aligned}
& d\boldsymbol{X}(t) = \boldsymbol{f}\big(\boldsymbol{X}(t), \boldsymbol{X}(t-\tau), t\big)\, dt + I\!H\big(\boldsymbol{X}(t), \boldsymbol{X}(t-\tau), t\big) \circ d\boldsymbol{W}(t), \\
& \qquad t_0 + \tau = t_1 < t \leqslant T < +\infty, \\
& d\boldsymbol{X}(t) = \boldsymbol{f}_0\big(\boldsymbol{X}(t), t\big)\, dt + I\!H_0\big(\boldsymbol{X}(t), t\big) \circ d\boldsymbol{W}(t), \\
& \qquad t_0 < t \leqslant t_1, \qquad \boldsymbol{X}(t_0) = \boldsymbol{X}^0.
\end{aligned}
$$

Hereinafter t is a time, t_0 and T are the initial and final times, $t_0 < T < +\infty$, $\tau > 0$ is a constant delay. The vector $\boldsymbol{X}(t) = \big(X_1(t), X_2(t), ..., X_n(t)\big)$ is the state vector in $\mathbb{R}^n$. The initial condition $\boldsymbol{X}^0 = \big(X_1^0, X_2^0, ..., X_n^0\big)$ is a second-order random vector distributed due to the probability density function (PDF), $p^0(\boldsymbol{x}), \boldsymbol{x} \in \mathbb{R}^n$. Its mean value is $\boldsymbol{m}_X^0 = \mathsf{E}\big[\boldsymbol{X}^0\big]$ and its covariance matrix is $I\!K_{XX}^0 = \mathsf{E}\big[(\boldsymbol{X}^0 - \boldsymbol{m}_X^0)(\boldsymbol{X}^0 - \boldsymbol{m}_X^0)^\intercal\big]$. The input $\{\boldsymbol{W}(t) = \big(W_1(t), W_2(t), ..., W_m(t)\big),\ t \geqslant t_0\}$ is the $\mathbb{R}^m$-valued Wiener stochastic process which consists of independent components and is independent of $\boldsymbol{X}^0$. A generalized derivative of $\boldsymbol{W}(t)$ with respect to t, denoted by $\{\boldsymbol{V}(t) =$

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$\big(V_1(t), V_2(t), ..., V_m(t)\big)$, $t \geqslant t_0\}$, is the vector Gaussian white noise with independent components such that

$$\mathsf{E}\big[\boldsymbol{V}(t)\big] = \boldsymbol{0}, \qquad \mathsf{E}\big[\boldsymbol{V}(t)\,\boldsymbol{V}^{\mathsf{T}}(t')\big] = I\!I_m\,\delta(t-t').$$

In these equations, $\boldsymbol{f}_0(\cdot,\cdot) = \big\{f_{0i}(\cdot,\cdot)\big\} : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$, $\boldsymbol{f}(\cdot,\cdot,\cdot) = \big\{f_i(\cdot,\cdot,\cdot)\big\} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$ and $I\!H_0(\cdot,\cdot) = \big\{h_{0ij}(\cdot,\cdot)\big\} : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n \times \mathbb{R}^m$, $I\!H(\cdot,\cdot,\cdot) = \big\{h_{ij}(\cdot,\cdot,\cdot)\big\} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n \times \mathbb{R}^m$ are given deterministic real continuous vector and matrix functions with respect to all arguments. E and T are the symbols of the mathematical expectation and the matrix transpositions, $I\!I_m$ is the identity matrix.

The scheme consists of several subschemes and includes:

(i) the adapted combination of the method of steps and expansion of the state space of the SODDE system, which transforms a non-Markov vector process into a chain of the Markov processes,

(ii) a procedure for constructing calculation formulae for obtaining values of moment functions for state vectors with increasing dimension on a given grid,

(iii) an algorithm for recalculating the initial conditions step by step for the specified functions.

For example, at step k the governing SODDE system will take the form as follows

$$\begin{aligned}
&d\boldsymbol{Y}(s) = \boldsymbol{0}\,ds, \quad \boldsymbol{Y}(0) = \boldsymbol{X}^0,\\
&d\boldsymbol{X}_0(s) = \boldsymbol{f}_0\big(\boldsymbol{X}_0(s), s_0\big)\,ds + I\!H_0\big(\boldsymbol{X}_0(s), s_0\big)\circ d\boldsymbol{W}_0(s),\ \ \boldsymbol{X}_0(0) = \boldsymbol{X}^0;\\
&d\boldsymbol{X}_1(s) = \boldsymbol{f}\big(\boldsymbol{X}_1(s), \boldsymbol{X}_0(s), s_1\big)\,ds + I\!H\big(\boldsymbol{X}_1(s), \boldsymbol{X}_0(s), s_1\big)\circ d\boldsymbol{W}_1(s),\\
&\qquad \boldsymbol{X}_1(0) = \boldsymbol{X}_0(\tau);\\
&\dots\ \ \dots\ \ \dots\ \ \dots\ \ \dots\ \ \dots\ \ \dots\\
&d\boldsymbol{X}_k(s) = \boldsymbol{f}\big(\boldsymbol{X}_k(s), \boldsymbol{X}_{k-1}(s), s_k\big)\,ds + I\!H\big(\boldsymbol{X}_k(s), \boldsymbol{X}_{k-1}(s), s_k\big)\circ d\boldsymbol{W}_k(s),\\
&\qquad \boldsymbol{X}_k(0) = \boldsymbol{X}_{k-1}(\tau),
\end{aligned}$$

where

$$\begin{gathered}
s \in [0,\tau], \quad t_q = t_0 + q\,\tau, \quad q = 0,1,2,...,N+1, \quad t_{N+1} \geqslant T, \quad s_q = t_q + s,\\
p_q(\boldsymbol{x}_q, s) = p_X(\boldsymbol{x}_q, s+t_q), \quad p_q(\boldsymbol{x}_q, 0) = p_{q-1}(\boldsymbol{x}_q, \tau), \quad p_0(\boldsymbol{x}, 0) = \overset{0}{p}_X(\boldsymbol{x}),\\
\boldsymbol{X}_q(s) = \boldsymbol{X}(s+t_q), \quad \boldsymbol{X}_q(0) = \boldsymbol{X}_{q-1}(\tau), \quad \boldsymbol{Y}(s) = \boldsymbol{Y} \equiv \boldsymbol{X}^0,\\
\boldsymbol{W}_q(s) = \boldsymbol{W}(s+t_q), \quad \boldsymbol{W}_q(0) = \boldsymbol{W}_{q-1}(\tau).
\end{gathered}$$

At this step, PDF $p_{Z_k}(\boldsymbol{z}_k, s)$ of the expanded state vector $\boldsymbol{Z}_k(s) = \mathrm{col}\big(\boldsymbol{Y}(s), \boldsymbol{X}_0(s), \boldsymbol{X}_1(s), ..., \boldsymbol{X}_k(s)\big)$ satisfies to the Fokker–Planck–Kolmogorov equation (FPK Eq.),

$$\frac{\partial p_{Z_k}}{\partial s} = I\!L_k\big[p_{Z_k}\big] \equiv -\sum_{i=1}^{n\,(k+2)} \frac{\partial\big(a_{ki}\,p_{Z_k}\big)}{\partial z_{ki}} + \frac{1}{2}\sum_{i,j=1}^{n\,(k+2)} \frac{\partial^2\big(b_{kij}\,p_{Z_k}\big)}{\partial z_{ki}\,\partial z_{kj}},$$

$$p_k(\boldsymbol{x}_k, ..., \boldsymbol{x}_1, \boldsymbol{x}_0, \boldsymbol{y}, 0) = p_{k-1}(\boldsymbol{x}_k, ..., \boldsymbol{x}_1, \boldsymbol{x}_0, \tau)\, \delta(\boldsymbol{y} - \boldsymbol{x}_0),$$

where

$$a_{ki} = f^+_{ki} + \frac{1}{2} \sum_{j=1}^{n\,(k+2)} \sum_{\ell=1}^{m\,(k+2)} \frac{\partial h^+_{ki\ell}}{\partial z_{kj}}\, h^+_{kj\ell}, \qquad b^+_{kij} = \sum_{\ell=1}^{m\,(k+2)} h^+_{ki\ell}\, h^+_{kj\ell},$$

$$\boldsymbol{f}^+_k(\boldsymbol{z}_k, s) = \left\{ \boldsymbol{f}^+_{k\ell}(\boldsymbol{z}_k, s) \right\} = \begin{bmatrix} \boldsymbol{f}^+_{k-1}(\boldsymbol{z}_{k-1}, s) \\ \boldsymbol{f}(\boldsymbol{x}_k, \boldsymbol{x}_{k-1}, s_k) \end{bmatrix},$$

$$I\!\!H^+_k(\boldsymbol{z}_k, s) = \left\{ h^+_{kij}(\boldsymbol{z}_k, s) \right\} = \begin{bmatrix} I\!\!H^+_{k-1}(\boldsymbol{z}_{k-1}, s) & \mathbf{0}_{n(k+1)\times m} \\ \mathbf{0}_{n\times m(k+1)} & I\!\!H(\boldsymbol{x}_k, \boldsymbol{x}_{k-1}, s_k) \end{bmatrix}.$$

Then if to exploit, for example, the classical fourth-order Runge–Kutta method to find approximations for $p_{Z_k}(\boldsymbol{z}_k, s)$ using FPK Eq., in the following form:

$$\widetilde{p}_{Z_k,\ell} = \widetilde{I\!\!L}_{k\ell}\big[\widetilde{p}_{Z_k,\ell}\big], \qquad \widetilde{I\!\!L}_{k\ell} = I\!\!I + \frac{h_{k\ell}}{6}\Big[\big(I\!\!L_{k\ell} + 4\, I\!\!L_{k,\ell+\frac{1}{2}} + I\!\!L_{k,\ell+1}\big) +$$
$$+ h_{k\ell}\,\big(I\!\!L_{k,\ell+\frac{1}{2}}\, I\!\!L_{k\ell} + I\!\!L^2_{k,\ell+\frac{1}{2}} + I\!\!L_{k,\ell+1}\, I\!\!L_{k,\ell+\frac{1}{2}}\big) +$$
$$+ \frac{h^2_{k\ell}}{2}\,\big(I\!\!L^2_{k,\ell+\frac{1}{2}}\, I\!\!L_{k\ell} + I\!\!L_{k,\ell+1}\, I\!\!L^2_{k,\ell+\frac{1}{2}}\big) + \frac{h^3_{k\ell}}{4}\, I\!\!L_{k,\ell+1}\, I\!\!L^2_{k,\ell+\frac{1}{2}}\, I\!\!L_{k\ell}\Big],$$

then it is possible to build direct formulae in symbolic form to compute approximations of different moment characteristics on grids with the steps $h_{k\ell}$.

References

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