Random Walks for Solutions of Some Elliptic Equations with Discontinuous Coefficients

A. S. Sipin[1], ***A. N. Kuznetsov***[1]

[1]Vologda State University, Vologda, cac1909@mail.ru, pmqqkan@gmail.com

We consider stochastic algorithms for solving the first boundary value problem for the equation

$$M_i u(x) = \mathbf{div}(\varepsilon_i(x) \cdot \mathbf{grad} u(x)) = 0, \quad x \in D_i, \quad i = 1, 2 \tag{1}$$

in a bounded domain $D \subset \mathbb{R}^m$, $\quad (m \geq 3)$, divided into two parts D_1 and D_2 with a common smooth boundary γ. The functions $\varepsilon_1(x)$ and $\varepsilon_2(x)$ have partial derivatives that are continuous in $\overline{D_1}$ and $\overline{D_2}$, respectively. They are bounded from below by the constant $\nu > 0$, and their normal derivatives on the boundary are consistent:

$$\lim_{y \in D_1, y \to x} \varepsilon_1(y) \frac{\partial}{\partial n} u(y) = \lim_{y \in D_2, y \to x} \varepsilon_2(y) \frac{\partial}{\partial n} u(y), x \in \gamma. \tag{2}$$

On the outer boundary Γ of the domain $D = D_1 \cup D_2$ the Dirichlet condition $u(x) = \varphi(x)$ is satisfied.

When constructing a statistical algorithm, the boundary value problem is replaced by an integral equation of the second kind with a substochastic kernel. The unbiased estimators of the solution are constructed on the trajectories of a homogeneous Markov chain with discrete time (random walk), the transition probability of which coincides with this kernel. For problem (1)-(2) with constants ε_1 and ε_2, the estimators are constructed on trajectories of Random Walk on Hemispheres in the papers Simonov [1], Ermakov and Sipin [2], Sipin [3].

In the case of a plane boundary γ, we use a Random Walk on Balls, which is based on formulas about the mean value for the solution $u(x)$ in the ball $T(x) \subset D$ centered at the point $x \in D$, the radius of which is R. Namely, let $T(x) \subset D_i$, $r = ||x - y||$, σ_m is full solid angle in $\mathbb{R}^m$. Then,

$$u(x) = \frac{1}{\sigma_m R(m-2)\varepsilon_i(x)} \int_T u(y) M_i v(y) dy = \int_T p_i(x,y) u(y) dy, \quad x \in D_i. \tag{3}$$

For $x \in \gamma$, we have

$$u(x) = \frac{2}{R\sigma_m(m-2)(\varepsilon_1(x) + \varepsilon_2(x))} \Big(\int_{T \cap D_1} M_1 v(y) u(y) dy + \\ + \int_{T \cap D_2} M_2 v(y) u(y) dy \Big) = \int_T p(x,y) u(y) dy, \tag{4}$$

where

$$v(y) = \int_r^R \left(\frac{1}{r^{m-2}} - \frac{1}{\rho^{m-2}} \right) d\rho.$$

There exists R_0 such that for $R < R_0$ the kernels of the integral operators in formulas (3) - (4) are stochastic.

To define the process of Random Walk on Balls, we take $\delta > 0$ and define Γ_δ and γ_δ as the δ-neighborhoods of the boundaries Γ and γ, respectively. Let the radius of the ball $T(x))$ is $R(x) = \min(R_0, dist(x, \Gamma))$. The transition from the current state to the next is carried out in accordance with the transition probability densities determined by formulas (3) - (4). The Random Walk continues in the region D_1 or in the region D_2 until it reaches Γ_δ or γ_δ. After that, there is a jump to the nearest point of the boundary Γ or γ, respectively. At the boundary Γ, the process ends and an estimator is calculated. It is equal to the value of $\varphi(x)$ at the last point of the walk. From the boundary γ, the walk returns to D_1 or to D_2, in accordance with the transition probability density $p(x, y)$ from formula (4).

The performance of the algorithm is ensured by the following properties:

1. The Random Walk on Balls in domain D_1 (in domain D_2) attains $\gamma_\delta \cup \Gamma_\delta$ in a finite number of steps. The average number of steps is finite.

2. The average number of visits by the process to the boundary γ_δ is finite.

In the case of a smooth boundary, the walk is constructed in a similar way.

Acknowledgement. This research was supported by Russian Science Foundation under grant 19-11-00020.

References

1. Simonov N.A. Monte Carlo methods for solving elliptic equations with boundary conditions containing the normal derivative Dokl. Math., 74 (2006), pp. 656-659.

2. Ermakov, S.M., Sipin, A.S. The walk in hemispheres process and its applications to solving boundary value problems. Vestnik St.Petersb. Univ.Math. 42, 155163 (2009).

3. Sipin, A.S. On Stochastic Algorithms for Solving Boundary-Value Problems for the Laplace Operator. J Math Sci 225, 812817 (2017)