On estimation of extremal index in queueing systems with mixture service times

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We study extremal behavior of waiting times and queue sizes in queueing systems with mixture service times.

Let $\{X_n, \ n \geq 1\}$ be a sequence of independent identically distributed (i.i.d) random variables (r.v.'s) with distribution function (d.f.) F. We define $M_n = \max(X_1, \ldots, X_n)$. It is obviously, that for d.f. of M_n the following equality holds:

$$\mathsf{P}(M_n < x) = F^n(x).$$

It's known that [7] if there exist the sequences $a_n > 0, \ b_n, \ n \geq 1$ such, that for non-degenerate distribution G the following relation

$$\mathsf{P}(a_n(M_n - b_n) \leq x) \to G(x), \quad n \to \infty, \tag{1}$$

holds, then d.f. F belongs to the maximum domain of attraction of d.f. G, $F \in MDA(G)$. Distributions G satisfying (1) are called *extreme value distributions* and have the following general form [3]:

$$P(X < x) = H(x) = \begin{cases} exp\left(-(1 + \eta \cdot \dfrac{x - \nu}{\sigma})^{-1/\eta}\right) & \eta \neq 0; \\ exp\left(-exp(- \ \dfrac{x - \nu}{\sigma})\right) & \eta = 0. \end{cases} \tag{2}$$

where $1 + \eta \cdot \dfrac{x - \nu}{\sigma} > 0$

For $\eta > 0$ we get Frechet distribution, if $\eta < 0$ Weibull distribution, if $\eta = 0$ Gunbel distribution.

If there exists sequence $u_n = u_n(x)$ such that $n\overline{F}(u_n) \to \tau$ as $n \to \infty$, then

$$\mathsf{P}(M_n \leq u_n(x)) \to e^{-\tau(x)}, \text{ as } n \to \infty, \ 0 < \tau(x) < \infty, \tag{3}$$

and conversely [7]. It's clear that $\tau(x) = e^{-x}$ for Gumbel distribution, $\tau(x) = x^{-\eta}$ for Frechet distribution and $\tau(x) = (-x)^{\eta}$ for Weibull distribution. The linear normalised sequence $u_n(x) = x/a_n + b_n$ is easy to find for some distributions.

In the case of dependent strictly stationary sequences additional conditions $D(u_n), D'(u_n)$ on mixing of r.v.'s (see [7]) ensure asymptotic extremal behaviour and relation (3) turns into

$$\mathsf{P}(M_n \leq u_n(x)) \to e^{-\theta\tau(x)}, \text{ as } n \to \infty, \ 0 < \tau(x) < \infty, \tag{4}$$

where $\theta \in [0,1]$ is called *extremal index* of $\{X_n\}$. While for independent sequence $\{\hat{X}_n\}$ associated with $\{X_n\}$ ($\hat{X}_n$ are independent r.v.'s with the same d.f. F) $\mathsf{P}(M_n \le u_n(x)) \to e^{-\tau(x)}$, as $n \to \infty$, where $\hat{M}_n = \max(\hat{X}_1, \dots, \hat{X}_n)$.

To estimate the extremal index θ block method can be used. Notice that (4) together with $n\overline{F}(u_n) \to \tau$ as $n \to \infty$ imply

$$\theta = \lim_{n\to\infty} \frac{\mathsf{P}(M_n \le u_n)}{n \log F(u_n)}. \tag{5}$$

Now we assume that $n = m \cdot h$, We divide sequence $X_1, \dots, X_N$ into m blocks of size h. For each block the maximum should be calculated,

$$M_{n,k} = \max(X_{(i-1)h+1}, \dots, X_{ih}),\ i = 1, \dots, m.$$

Denote $N(u_n) = \#(i \le n, X_i > u_n)$ – the number of exceedances of u_n by $X_1, \dots, X_n$ and $m(u_n) = \#(k \le n, M_{n,k} > u_n)$ – the number of blocks with one or more exceedances, then

$$\hat{\theta} = \frac{1}{h} \cdot \frac{\log\left(1 - \dfrac{m(u_n)}{m}\right)}{\log\left(1 - \dfrac{N(u_n)}{n}\right)} \tag{6}$$

is the estimate of extremal index θ [3].

Now we extend results of extreme value theory to mixture distributions. Let X has *m-component (positive) mixture distribution*, in the following form [10]

$$F_X(x) = p_1 F_{Z_1}(x) + \dots + p_m F_{Z_m}(x),\ \sum_{k=1}^{m} p_k = 1,\ p_k \ge 0, \tag{7}$$

where components of mixture Z_k has d.f. F_{Z_k}. The following relation gives the limiting distribution of maximum of m-component mixture distribution under linear normalization [1]. Let $X_1, \dots, X_n$ be i.i.d. r.v.'s with common d.f. $F(x)$ in the form (7) and $M_n = \max(X_1, \dots, X_n)$. Then

$$\mathsf{P}(M_n \le u_n(x)) = F^n(u_n(x)) \to \prod_{k=1}^{m} (H_k(x))^{p_k} \text{ as } n \to \infty \tag{8}$$

if and only if

$$F_{Z_k}(u_n(x)) \to H_k(x) \text{ as } n \to \infty,$$

where $u_n(x)$ -linear normalized sequence and $H_k(x)$ are non-degenerate d.f.'s.

In this work we estimate extremal index of waiting times and queue size processes in the queueing systems with mixture service times (7). The estimation is based on regenerative simulation [8] and relation (6).

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