Asymptotically optimal arrangements of service stations for different optimality criteria

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Often, a necessary part of the maintenance is to move to the desired point in space: ambulances, taxi cabs, repair crews must come on call to a patient or client. The specified transportation process begins from one of the base stations: in in these cases, they are a hospital, a taxi company and a management building.

The closer the base station is placed to the call point, the higher the service rate of the call. Therefore the easiest way to improve the quality of work is to increase the number of base stations, but this is an extensive option, fraught with difficulties of various kinds, in particular, financial; starting with some moment it not only ceases to be optimal, but also does more harm than good.

An intensive option in the matter of increasing the speed of call service involves searching a rational arrangement of stations and leads us to the problem of optimal placement service stations on the set; but since the tasks of finding the best placement are very difficult, in practice, they often make a controlled error and seek only an approximate (asymptotic) solution.

Let us give a brief overview of the problem: given a set, from some points of which calls can be made. To handle calls, the set contains a number of stations. Each station serves calls from those points for which it is closest. The problem of finding an asymptotically optimal arrangement is to indicate such an algorithm for the placement **of an arbitrary** number of stations, so that as their number increases, the placement quality approaches ideal.

Definition 1. The arrangement *of n stations on a line is a set*

$$x = \{x_1, x_2, \dots, x_n\}$$

of the points where the stations are located.

Definition 2. The influence region *of the station x_i is a set C_i of points, for which this station is the nearest one*

$$C_i = \{v \in \mathbb{R} : |v - x_i| \leqslant |v - x_j|, j = 1, 2, \dots, n\}.$$

The coordinate of the call is the random variable ξ with the probability density function ρ. To assess the optimality of the stations arrangement we choose **a criterion** φ:

$$\varphi(x) = \mathbb{E} \min_{1 \leqslant i \leqslant n} |\xi - x_i|^s, s > 0. \quad (1)$$

Definition 3. *The arrangement* $x^* = \{x_1^*, x_2^*, \ldots, x_n^*\}$ *is* **optimal** *among the set of arrangements* X*, if*

$$\varphi(x^*) = \min_{x \in X} \varphi(x).$$

Definition 4. *The arrangement* x *is* k**-order asymptotically optimal** ($k \in \mathbb{N}$)*, if*

$$\lim_{n \to \infty} n^{s+k-1}(\varphi(x) - \varphi(x^*)) = 0,$$

here x^* *is the optimal arrangement.*

Most of the studies devoted to this problem (for example, [1]) are limited to considering a single value $s = 1$ of the exponent in the optimality criterion. This case corresponds to minimizing the average distance between the call coordinate and the nearest service station. This is the most classic example of a scoring metric and is the easiest to learn, but it makes the solution less flexible: for example, if it is important to prevent calls that are too far from the nearest station, this can be achieved by having additional penalties for long distances, that is, choosing $s > 1$; if it is more important to improve processing areas of high call density, you can lower the distance penalty by choosing $s < 1$.

In [2] the authors presented a second-order asymptotically optimal algorithm for placing service stations on a segment for the case $s \in \mathbb{N}$.

The latest research has resulted in an algorithm of the same optimality order for an arbitrary $s > 0$ and an asymptotically optimal algorithm for a special class of metric spaces. Different metrics reflect the features of real conditions: the terrain, the presence of highways and the quality of the road surface.

We denote $M = \max(\rho(u))$, $m = \min(\rho(u))$.

We choose the numbers a and b in such a way that the conditions $0.5 < a \leqslant b < 1$ and $3a - b > 1$ are fulfilled. For example, $a = 0.6$ and $b = 0.7$.

We divide the initial segment G into disjoint segments G_i, $i = 1, 2, \ldots, k$ ($k = [n^b]$) such that for each admissible i the following conditions are fulfilled:

$$G_i \leqslant \frac{G}{n^a}, \qquad G_i \geqslant \frac{M^{1/(s+1)} G}{m^{1/(s+1)} n}. \tag{2}$$

The numbers n_i, defined by the relations

$$n_i = \frac{\int\limits_{G_i} \rho^{1/(s+1)}(u)\, du}{\int\limits_{G} \rho^{1/(s+1)}(u)\, du} \cdot n, \tag{3}$$

must be natural.

The last condition can be achieved by choosing the number n sufficiently large, because of the continuity of the integral and the condition $\sum\limits_{i=1}^{k} n_i = n$.

We note that $n_i \geqslant 1$ for all admissible i. Next, we divide each G_i into n_i equal segments Δ_i and we put n_i stations into their centers.

This sequence of arrangements is second-order asymptotically optimal.

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References

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