On characteristic function estimation for weighted sums of random vectors

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Let $X, X_1, X_2, \ldots, X_n$ be a sequence of independent identically distributed random variables with zero mean, unit variance and finite third absolute moment. It follows from the Berry–Esseen inequality

$$\sup_{x\in R} |P(\frac{1}{\sqrt{n}}\sum_{j=1}^{n} X_j \le x) - \Phi(x)| \le C\frac{\mathbb{E}|X|^3}{\sqrt{n}},$$

that the rate of convergence of the normalized sum to the standard normal distribution is of order $O(\frac{1}{\sqrt{n}})$. However, it was shown by Klartag and Sodin [1], that if we consider weighted sums of random variables with finite fourth moment

$$\sum_{j=1}^{n} \theta_j X_j,$$

where $\theta = (\theta_1, \theta_2, \ldots, \theta_n) \in Q = \{\theta_1, \theta_2, \ldots, \theta_n : \sum_{j=1}^{n} \theta_j^2 = 1\}$, then the order of the convergence rate can be significantly improved for most weights coefficients. It was shown that

$$\mathbb{E}_\theta \sup_{x\in R} |P(\sum_{j=1}^{n} \theta_j X_j \le x) - \Phi(x)| \le C\frac{\mathbb{E}X^4}{n}$$

In order to generalize this result to the multidimensional case for all Borel convex sets, we follow Bhattacharya and Rao Ranga [2] and make use of truncated random vectors. Define Y_j and Z_j as

$$Y_j = X_j 1_X(\|\theta_j X_j\| \le 1),$$

$$Z_j = Y_j - \mathbb{E}Y_j,$$

for $j = 1, \ldots, n$, where 1_X is the indicator function. And also define characteristic function of the random vector Z_j as follows

$$\varphi_j(t) = \mathbb{E}\exp(itZ_j),\ j = 1, \ldots, n.$$

Based on the approaches of Klartag and Sodin [1] and the technique of Bhattacharya and Rao Ranga [2], we proved the following Lemma

Lemma 1. *Let $X_1, X_2, \ldots, X_n$ be independent random vectors of dimension k with zero mean $\mathbb{E}X_j = \overline{0}$, unit covariance matrix $cov(X_j) = I$ and finite fourth absolute moment $\delta_j^4 = \mathbb{E}\|X_j\|^4$ for $j = 1, \ldots, n$. Then there exist a subset $\Im \subseteq Q$ with normalized Lebesgue measure $\lambda(\Im) \geq 1 - C_1(\alpha, k)\exp(-c_1 \frac{n}{\delta^4})$ such that for any $\theta \in \Im$ with $\delta_\theta^4 \leq \frac{1}{8k}$ and for any vector α with integer non-negative components, $|\alpha| \leq k+1$, one has*

$$\int\limits_{\|t\| \leq \frac{n}{\delta^4}} \Big| D^\alpha (\prod_{j=1}^{n} \varphi_j(\theta_j t) - \phi_{0,V})(t) \Big| dt \leq C(k,\alpha)\Big(\delta_\theta^4 + \frac{\delta^4}{n} + \sum_{|\nu|=3} \Big| \sum_{j=1}^{n} \theta_j^3 \mu_\nu \Big|\Big),$$

where $\delta_\theta^4 = \sum_{j=1}^n \theta_j^4 \delta_j^4$, $\delta^4 = \frac{1}{n}\sum_{j=1}^n \delta_j^4$, $\mu_\nu = \mathbb{E}Z_j^\nu$, $V = \sum_{j=1}^n \theta_j^2 cov(Z_j)$, $\phi_{0,V}$ is the characteristic functions of normal random vector with zero mean and covariance matrix V and D^a is a differential operator.

References

1. B. Klartag, S. Sodin, Variations on the Berry – Esseen theorem, *Theory of Probability and its Applications*, **56**:3 (2011) 514-533.

2. R. Bhattacharya, R. Rao Ranga, *Normal Approximation and Asymptotic Expansions*, New York, Wiley, 1976.