On statistical stability analysis of discrete-time Markov chains

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In this talk we address the problem of instability detection of a queueing system modelled using a continuous-time Markov chain. As a theoretical base, we use the results of Mandjes et al. [1]. Below we briefly explain these results.

Consider a collection of parameter-dependent queuing models $\{X_k^\lambda\}_{k\geq 0}$ in the form of the discrete-time Markov chains living in the state $\mathcal{X}$ and depending on some internal (real, constant) parameter, say, input or service rate, defined in a finite set $\lambda \in \mathcal{L}$. The task is to find the set of model instability $\overline{\mathcal{L}} \subset \mathcal{L}$ by obtaining its representative, $\lambda^* \in \overline{\mathcal{L}}$. The instability is checked using the monotone Lyapunov function $f : \mathcal{X} \to [0, \infty)$ is used. It is assumed that f is unbounded, that is,

$$\liminf_{|x|\to\infty} f(x) = \infty,$$

and its increments are (uniformly) bounded by some constant ϕ:

$$|f(X_{k+1}^\lambda) - f(X_k^\lambda)| \leq \phi, \quad k \geq 0.$$

In the queueing environment, f may be e.g. the queue size of the model.

To obtain λ^*, a modified simulated annealing algorithm is adopted. A sequence $\{\lambda_i\}_{i\geq 1}$ of parameter values is constructed in such a way to maximize the system instability (the drift of Lyapunov function for the model with the given parameter). As such, the system trajectory is a chain of connected trajectories of finite length obtained by finite simulation runs with corresponding parameter values. The trajectory is built up in iterations of the algorithm. At kth iteration the initial value of the chain is taken as Y_{k-1}, with some (random) Y_0. Then two independent trajectories of the Markov chain are sampled for a number of steps $\tau(Y_{k-1})$, where τ is a linear function, using the (new) randomly sampled from $\mathcal{L}$ parameter value λ and the old value λ_{k-1}. Finally, the new state, $Y_k = X^\lambda_{\tau(Y_{k-1})}(Y_{k-1})$ and the new parameter value, $\lambda_k = \lambda$ are selected with acceptance probability

$$\exp\left(\eta \min\left[f(X^\lambda_{\tau(Y_{k-1})}(Y_{k-1})) - f(X^{\lambda_{k-1}}_{\tau(Y_{k-1})}(Y_{k-1})), 0\right]\right),$$

where η is a small positive constant (in the notation, the dependence on the initial value of the chain is stressed).

Based on the trajectory obtained, a closed family of random variables $Z(w)$, which stochastically majorities the increment $f(Y_k)$ at the kth step, is constructed. A new Markov chain, W_k, $k \in \mathbb{Z}_+$, is built using a stochastic recursion:

$$W_k = W_{k-1} + Z(W_{k-1}), \quad W_0 = 0.$$

Intuitively, such a sequence describes the “largest possible value” for the Lyapunov function increment given the system is stable for all the parameter values in $\mathcal{L}$. Overshooting this boundary, in stochastic sense, is likely only for non-stationary systems. More precisely, if $q_k^{(\alpha)}$ is the upper quantile of level α for the random variable W_k, and $f(Y_k) > q_k^{(\alpha)}$, then we accept the hypothesis of non-stationarity of the original Markov chain. Otherwise, we assume that the data are insufficient to draw a conclusion about the system stability.

The following distribution is used to build the stochastic recursion for $\{W_k\}_{k \geq 0}$:

$$P(Z(w) \geqslant z) = \begin{cases} \min\left[1, e^{-\frac{(z-\alpha_1(w))^2}{2\alpha_2(w)}} + n(w)e^{-\frac{(z-\alpha_3(w))^2}{2\alpha_4(w)}}\right], & \text{if } z > 0, \\ 1, & \text{otherwise}, \end{cases}$$

where:

$$\begin{aligned} \alpha_1(w) &= \sigma\phi - \sigma n(w)\delta, & \alpha_2(w) &= (\phi+\delta)^2\sigma^2 n(w), \\ \alpha_3(w) &= \sigma\phi - w + \kappa, & \alpha_4(w) &= \phi^2\sigma^2 n(w). \end{aligned}$$

In these formulas:

- $\tau(w)$ is a simulation time;
- $n(w)$ is the smallest integer such that $\sigma n(w) \geqslant \tau(w)$;
- κ is a lower bound for the modulus of the initial state of the chain;
- σ is a lower bound for the number of steps;
- δ is a negative drift that a process must demonstrate to be stationary.

In our talk we demonstrate an application of this method to the task of multiclass multiserver queueing system stability region detection.

References

1. M. Mandjes, B. Patch, N. S. Walton, Detecting Markov Chain Instability: A Monte Carlo Approach *Stochastic Systems 7(2)* **383** (2017) 289-314.