Structural Stability of Financial Market Models

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The main aim of the present paper is to demonstrate the importance of structural stability for financial modelling, in particular, discuss its relation to the continuity and approximation properties of superhedging prices. The structural stability is the fundamental property of a model, which means that the qualitative behaviour of the model is unaffected by small (in a certain sense) perturbations of its dynamics. The term “structural stability” is borrowed from dynamical systems theory. Structurally stable systems were introduced in 1937 by Andronov and Pontryagin [1] under the name “systèmes grossiers,” or coarse systems. From an economic point of view, such a qualitative behaviour of the model of the financial market is to admit no “arbitrage”, in some sense to be made precise.

Literature overview. The idea of stability of no arbitrage property under some perturbation of the model is, of course, present in the literature. We restrict ourselves to mention only few related papers. Let us start with Schachermayer [2], where a “robust no-arbitrage condition” was introduced in the case of market friction (proportional transaction costs). The economic meaning of this notion is that there is still room for the broker to offer some discount in quoting bid and ask prices without creating an arbitrage possibility. There are also ad hoc definitions of robust no-arbitrage property (in the framework of a particular model) e.g. in Bayraktar, Zhang and Zhou [3], considering the case with non-tradable options that are quoted with bid-ask spreads. In this set-up, the robust no-arbitrage property turns out to be equivalent to no-arbitrage under the additional assumption that hedging options with non-zero spread are non-redundant. Sometimes such kind of properties are used implicitly, as in Hou and Obłój [4], considering a continuous time model of financial market with primary assets, options that are non-tradable except initial time moment and continuously traded European options. The general duality results of the paper exploit an assuption, concerning these options, ensuring that their prices are not “on the boundary of the no-arbitrage region”, i.e., calibrated martingale measures exist under arbitrarily small perturbations of the initial prices. For the frictionless market a simple, but nice result is presented in Ostrovski [5], where it is shown that for non-redundant one-step model no arbitrage property is preserved under sufficiently small perturbation of the initial probability in the total variation metric. Thus, *depending on the context, the structural stability can be formalised in different ways.*

We propose a formalisation of structural stability in the framework of a *Guaranteed Deterministic Approach* (GDA), developed in a series of papers by the author. The main premise of GDA is based on a specific assumption concerning a priori information regarding price movements. The corresponding market model formalise the “uncertainty” of price dynamics with discrete time and can be considered as purely deterministic: the initial problem setting in the GDA framework does not use any probability or family of probabilities, see Smirnov [6].

Concerning the superheghing problem within the GDA framework, we should stress that we adopt an alternative interpretation to the common robust approach to pricing of contingent liabilities. Our interpretation is game-theoretic: we deal with a deterministic dynamic two-player zero-sum game of “hedger” against “market.”[1] A family of probabilities appears as a secondary notion, thanks to the introduction of mixed strategies of the “market”[2]; this this makes it possible to use the mathematical techniques based on the game equilibrium[3] instead of the usual duality method.

Formally, from the contemporary point of view, the guaranteed deterministic approach to the superhedging problem can be classified as a specific path-wise (or pointwise) approach addressing uncertainty in market modelling by defining a set of deterministic market scenarios , a result of agents beliefs. Or it can be formally described in terms “quasi-sure” approach[4], by the choice of a collection of probabilistic models (possible priors) for the market. We share an idea, suggested in unpublished work of L. Carassus and T. Varioglu about 15 years ago and finally published in Carassus and Vargiolu [12]: in order to get a meaningful theory, it is reasonable to assume the boundedness of price increments. One of the first publications to develop a kind of GDA is an article published in 1998 by V. Kolokoltsov[5], see Kolokoltsov [13]. To the best of our knowledge, this was the first work to explicitly articulate this approach to pricing and hedging contingent claims. The GDA is closely related to a class of market models called interval models in Bernhard, Engwerda, Roorda, Schumacher, Kolokoltsov, Saint-Pierre and Aubin [14], especially to the ideas and results of Kolokoltsov published in Chapters 11–14 of this book, including independent discovery of the *game-theoretic interpretation of risk-neutral probabilities under the assumption of no trading constraints*; we find this interpretation to be quite important from an economic point of view.

Financial market model. Let us describe shortly a financial market

[1] A related formulation of the upper hedging price based on the game-theoretic probability is present in Matsuda and Takemura [7].

[2] However, this material (concerning the corresponding mixed strategies) is not needed for the present paper; interested readers can refer, for example, to Smirnov [8].

[3] In particular, we use our result in Smirnov [9].

[4] We refer to Bouchard and Nutz [10], and to Burzoni, Frittelli, Hou, Maggis, and Obłój [11] for these two robust modelling approaches and for detailed review of large literature focusing on robust approach to mathematical finance.

[5] The guaranteed deterministic approach was developed by us in late 90-s (although at that period we were not aware of Kolokoltsov's paper), but published (primarily in Russian) only in the last three years, together with some recent new results.

model in the GDA framework. Consider discounted prices of n risky assets; without loss of generality, we can suppose that a risk-free asset has a fixed price equal to one, and so in what follows we call "discounted prices" simply "prices." Let X_t be the price vector at time t and $\Delta X_t = X_t - X_{t-1}$ be the vector of price increments. The above-mentioned assumption about a priori information describing price movements is as follows: the price increments ΔX_t lie in a priori given (non-void) closed set sets[6] $K_t(\cdot) \subseteq \mathbb{R}^n,\ t = 1, \ldots, N$. Denote by B_t the set of possible trajectories (or paths) of asset prices in the time interval $[0, t]$, i.e.

$$B_t = \{(x_0, \ldots, x_t) : x_0 \in K_0, \Delta x_1 \in K_1(x_0), \ldots, \Delta x_t \in K_t(x_0, \ldots, x_{t-1})\}.$$

We suppose that there are trading constraints, concerning only risky assets. These are described by a priori given sets $D_t(\cdot) \subseteq \mathbb{R}^n,\ t = 1, \ldots, N$, depending on price prehistory, which are assumed to be convex and such that $0 \in D_t(\cdot)$. An admissible hedging strategy at time step t is therefore $h \in D_t(\cdot)$.

Bellman–Isaacs equations for the superhedging problem. To apply the GDA to the described above financial market model with trading constraints, we set the superhedging problem of a contingent claim on the American option, based on dynamic programming principle. Let us denote by $v_t^*(\cdot)$ the infimum of the portfolio value at time t that guarantees, given the price history, a choice of an appropriate hedging strategy covering current and future liabilities due to possible payments on the American option. The corresponding Bellman–Isaacs equations can be derived directly based on the economic sense, by choosing at step t the "best" admissible hedging strategy $h \in D_t(\cdot) \subseteq \mathbb{R}^n$ for the "worst" scenario of (discounted) price increments $y \in K_t(\cdot)$ for the given payoff functions $g_t(\cdot)$, describing the potential payouts on the option. Thus, we obtain the recurrence relation (see Smirnov [6]), in fact based on dynamic programming principle, which is the starting point of studying the superhedging problem within GDA framework and can also be regarded as a kind of axiom: for $t = N, \ldots, 1$

$$\begin{gathered} v_{t-1}^*(\bar{x}_{t-1}) = g_{t-1}(\bar{x}_{t-1}) \vee \inf_{h \in D_t(\bar{x}_{t-1})} \sup_{y \in K_t(\bar{x}_{t-1})} \left[v_t^*(\bar{x}_{t-1}, x_{t-1} + y) - hy \right], \\ v_N^*(\bar{x}_N) = g_N(\bar{x}_N), \end{gathered} \tag{1}$$

where $\bar{x}_{t-1} = (x_0, \ldots, x_{t-1})$ represents the price history up to the present moment $t - 1$, the symbol $\vee$ denotes maximum, and $hy = \langle h, y \rangle$ is the dot product of the vectors h and y. In (1), the functions v_t^* and the corresponding suprema and infima take values in an extended set of real numbers $\mathbb{R} \cup \{-\infty, +\infty\} = [-\infty, +\infty]$, which is the two-point compactification[7] of $\mathbb{R}$. In

[6] The dot "$\cdot$" indicates the variables representing the price evolution. More precisely, it indicates the prehistory $\bar{x}_{t-1} = (x_0, \ldots, x_{t-1}) \in (\mathbb{R}^n)^t$ for K_t, and it indicates the history $\bar{x}_t = (x_0, \ldots, x_t) \in (\mathbb{R}^n)^{t+1}$ for the functions v_t^* and g_t introduced below.

[7] The neighborhoods of points $-\infty$ and $+\infty$ are given by $[\infty, a)$, $a \in \mathbb{R}$ and $(b, +\infty]$, $b \in \mathbb{R}$, respectively.

a certain sense, this approach can be regarded as quite general: neither measurability conditions nor "no arbitrage" assumptions are initially imposed. It is convenient to assume (formally) that $g_0 \equiv -\infty$ (there are no liabilities to pay at the initial time); $g_t \geq 0$ for $t = 1, \ldots, N$ in the case of an American option.[8] Multivalued mappings $x \mapsto K_t(x)$ and $x \mapsto D_t(x)$, in addition to functions $x \mapsto g_t(x)$, are assumed to be defined for all $x \in (\mathbb{R}^n)^t$, $t = 1, \ldots, N$. Therefore, functions $x \mapsto v_t^*(x)$ are defined by (1) for all $x \in (\mathbb{R}^n)^t$, $t = 1, \ldots, N$.

Relevant "no arbitrage" notions. The different notions of "no arbitrage" in the framework of the deterministic market model can be relevant[9] and were introduced in Smirnov [16]; in this paper the corresponding geometric criteria were obtained. For the convenience of the reader, we give several definitions of the notions.

Definition 1. By deterministic arbitrage opportunity (DAO) at time step t, we mean that there exists a strategy $h^* \in D_t(\cdot)$ such that $h^* y \geq 0$ for all $y \in K_t(\cdot)$ and there exists a price movement $y^* \in K_t(\cdot)$ such that $h^* y^* > 0$. By deterministic sure arbitrage (DSA) at time step t, we mean that there exists a strategy $h^* \in D_t(\cdot)$ such that $h^* y > 0$ for all $y \in K_t(\cdot)$. We say that there is deterministic sure arbitrage with unlimited profit (DSAUP) at time step t if the function $h \mapsto \inf\{hy,\ y \in K_t(\cdot)\}$ takes arbitrarily large values for $h \in D_t(\cdot)$.

Using these three notions of "arbitrage", we can define the corresponding "no arbitrage" properties on a time interval: each "no arbitrage" property on a time interval is tantamount to the corresponding "no arbitrage" property at every time step of this interval for any price prehistory. So, we consider the following "no arbitrage" properties: no deterministic arbitrage opportunity (NDAO), no deterministic sure arbitrage (NDSA), and no deterministic sure arbitrage with unbounded profit (NDSAUP)[10].

According to our interpretation, $K_t(\cdot)$ reflects the agent's beliefs about price movements, which are naturally inexact; on the other hand, the trading constraints are supposed to be defined exactly. As we have mentioned above, within DSA framework we formalise *uncertainty of price movement*, so a reasonable model should satisfy the following "uncertainty principle"[11]. Consider

[8] European or Bermudian options can be also considered using (1): if no payment is due at a moment of time t, we formally set $g_t \equiv -\infty$.

[9] Two notions of arbitrage introduced below, DAO (deterministic arbitrage opportunity) relates to "One Point Arbitrage" and in our setting is also equivalent to quasi-sure arbitrage of Bouchard and Nutz [9], while DSA (deterministic sure arbitrage) relates to "Strong Arbitrage", to use the unified terminology of robust modelling in Burzoni, Frittelli, Hou, Maggis, and Obłój [10]. A detailed analysis of the relation between different "no-arbitrage" notions in the framework for robust modelling of financial markets in discrete time is presented in Obłój and Wiesel [15].

[10] Note that in the case of conic trading constraints (in particular, in the case of no trading constraints, i.e. $D_t(\cdot) \equiv \mathbb{R}^n$) NDSAUP coincide with NDSA.

[11] Which is not the case for the traditional model with the reference probability, where the universe containing all states of the world is fixed (up to a set of zero measure) by means of considering equivalent measures.

the trading constraints $D_t(\cdot)$ as fixed; if there is no "arbitrage" (in a certain sense) for the initial model with price dynamics described by $K_t(\cdot)$, then for the model with price dynamics described by $\tilde{K}_t(\cdot) \supseteq K_t(\cdot)$ the corresponding "no arbitrage" condition is also valid. The meaning is rather clear: if we cannot realise "arbitrage" for initial model, then all the more we cannot realise "arbitrage" for a model with more uncertainty, i. e. with less information about price movement. Note that NDSA and NDSAUP satisfy the uncertainty principle, whereas NDAO, in general, do not satisfy this principle and therefore can be considered as not relevant for the GDA framework (however, NDAO can satisfy this principle if certain additional assumptions are admitted, namely "robustness" property, defined below).

Structural stability (robustness of "no arbitrage"). Hereinafter, we assume that all the sets $K_t(\cdot)$ are bounded (hence compact), unless otherwise stated. Since our interpretation appeals to a vague knowledge about the price behaviour, we have introduced an important concept of structural stability. In our context it is formalised as follows: a specific "no arbitrage" property (which reflects the qualitative behaviour of a price dynamics) is unaffected by perturbations of $K_t(\cdot)$ that are sufficiently small with respect to the Pompeiu–Hausdorff metric d_{PH}; we call such a "no arbitrage" property robust[12] or coarse. In what follows, we consider two coarse "no arbitrage" properties: robust no deterministic arbitrage opportunity (RNDAO) and robust no deterministic sure arbitrage with unlimited profit (RNDSAUP). More precisely it is defined below.

Definition 2. Suppose that the initial market model satisfies the NDAO (respectively NDSAUP) condition and consider a perturbed model with increments ΔX_t lying in compact sets $\tilde{K}_t(\cdot)$ for $t = 1, \ldots, N$. The robust (or coarse) NDAO (respectively NDSAUP) property, abbreviated RNDAO (respectively RNDSAUP), means that for any $t \in \{1, \ldots, N\}$ and any prehistory of the prices $x \in B_{t-1}$, there exists $\epsilon_t(x) > 0$ such that if $d_{PH}(\tilde{K}_t(x), K_t(x)) \leq \epsilon_t(x)$, then the perturbed market model still satisfies the NDAO (respectively NDSAUP) condition[13].

The corresponding geometric criteria (in terms of convex hulls of $K_t(\cdot)$) were obtained in Smirnov [16] and Smirnov [17]. Here we mention only two of them, which are of interest in the simple but important for applications case of no trading constraints, i. e. when $D_t(\cdot) \equiv \mathbb{R}^n$. The condition NDAO is equivalent to the Jacod–Shiryaev geometric criterion:[14] $0 \in \mathrm{ri}(\mathrm{conv}(K_t(\cdot)))$, $t = 1, \ldots, N$, and RNDAO is equivalent to the "enhanced" Jacod–Shiryaev geo-

[12] We feel now that the term "robust" is overused in the literature (with different meanings) and the term "coarse" would be better, but unfortunately "robust" is already used in our papers.

[13] In fact, it can be weaken as follows: the convex hull of $\tilde{K}_t(x)$ need to be close to the convex hull of $K_t(x)$, i.e. $d_{PH}(\mathrm{conv}(\tilde{K}_t(x)), \mathrm{conv}(K_t(x))) \leq \epsilon_t(x)$. Note that $d_{PH}(\mathrm{conv}(A), \mathrm{conv}(B)) \leq d_{PH}(A, B)$ for compact sets A and B.

[14] In the probabilistic setting, this geometric criterion (understood almost surely) was found by Jacod and Shiryaev [18].

metric criterion[15] $0 \in \operatorname{int}(\operatorname{conv}(K_t(\cdot)))$, $t = 1, \ldots, N$, where $\operatorname{conv}(A)$ is the convex hull of a set A, $\operatorname{ri}(A)$ is the relative interior of a convex set A, and $\operatorname{int}(A)$ is the interior of a set A.

Continuity of pricing. We argue that the continuity property of superhedging price are related to the structural stability.

Theorem 1. Suppose that the robust condition of no arbitrage opportunities RNDSAUP holds, for $s = 1, \ldots, N$, the functions of potential payments g_s are continuous and multivalued mappings $\bar{x}_{s-1} \mapsto K_s(\bar{x}_{s-1})$ are h-continuous (continuity with respect to the Pompeiu–Hausdorff metric[16]). Then, the functions v_s^*, defined by (1) with $D_s(\cdot) \equiv \mathbb{R}^n$, are uniformly continuous and bounded on B_s; moreover, the continuity modulus of functions v_s^* can be estimated with the help of recurrent inequalities.

This result was obtained in Smirnov [19] in the case of no trading constraints[17]; in general case Theorem 1 was obtained in Smirnov [8].

Let us introduce a related probabilistic model, assuming that there are no trading constraints. Suppose that the initial price takes values in some compact set $F_0 = K_0$. Denote $F_t(x) = x + K_t(x)$, $x \in B_{t-1}$, $t = 1, \ldots, N$. Consider a family of multifunctions $F_t(\cdot)$, $t = 1, \ldots, N$ and a family $\mathcal{P}_t$ of the stochastic kernels P_t, $t = 1, \ldots, N$; the measure $P_t(\bar{x}_{t-1}, \cdot)$ is interpreted as a conditional distribution of X_t given the price prehistory $\bar{X}_{t-1} = \bar{x}_{t-1} \in B_{t-1}$. A measure P_0 is interpreted as a (marginal) distribution of X_0. The corresponding probability $\mathbb{P}$ can be defined using Ionescu Tulcea construction.

We say that the consistency condition (the relation between stochastic and deterministic models) is satisfied if

$$\operatorname{supp}(P_t(x, \cdot)) = F_t(x), \quad x \in B_{t-1},\ t = 1, \ldots, N; \qquad \operatorname{supp}(P_0) = F_0. \tag{2}$$

We prove in Smirnov [20] the following result.

Theorem 2. In the case of no trading constraints, under the same assumptions as in Theorem 1, the superhedging prices for the American option in the usual probabilistic setting, satisfying the consistency condition (2), coincide almost surely[18] with the corresponding solutions of (1).

As a consequence, under assumptions of Theorem 2 the superhedging prices for the usual probabilistic setting admit a continuous version.

We have noticed, that Proposition 3.7 in the Section 3.2 of Carassus, Obłój and Wiesel [21] about the continuity (as a function of price history) of superhedging price is not valid unless one of its assumptions is strengthen. Using our terminology we can formulate the result of this proposition in equivalent form as follows. The assumptions of Proposition 3.7 the paper mentioned above are

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[15] Note that in general RNDSAUP does not imply $\operatorname{int}(\operatorname{conv}(K_t(\cdot))) \neq \emptyset$.

[16] For compact-valued mappings, h-continuity is equivalent to continuity.

[17] Note that in this case RNDSAUP is equivalent to RNDAO.

[18] Under quite general assumptions GDA pricing is not less the probabilistic pricing (almost surely) if consistency condition holds and there are simple examples where it is strictly greater.

the following: The initial price is assumed to be fixed, $X_0 = x_0$; the correspondences $F_t(\cdot)$ are compact-valued and uniformly[19] continuous; the payoff function g on the European option is continuous; no quasi-sure arbitrage condition[20] with respect to the $\mathcal{P}_t$, the set of priors at time t, which consists of all the kernels, satisfying (2), holds for $t = 1, \dots, N$. The proposition in question states that under these assumptions the quasi-sure superhedging price $V_t,\ t = 0, \dots, N$ coincides with the almost sure superhedging price, and the functions $V_t = v_t(\bar{X}_t)$ are continuous[21] for $t = 1, \dots, N$.

We have constructed a counterexample clarifying the origin of the mistake in this assertion and showing, that NDAO condition is still insufficient for the continuity of the superhedging price. Note that Proposition 3.8 in Carassus, Obłój and Wiesel [21] is nevertheless valid, since the condition RNDAO is fulfilled.

Model approximation and structural stability. For the original market model, a natural way of solving the problem approximately is to construct a perturbed market model such that compacta describing the uncertainty of price movement have simple structures (e.g., they could be finite sets). To preserve the economic meaning of the solution to the problem for the perturbed market model (which is to have qualitative properties similar to those of the original system), we must preserve the structural stability conditions. If using the perturbed market model we obtain a numerical solution with the prescribed error and such that the price increments lie in the compacta $\tilde{K}_t(\cdot)$, the meeting of condition RNDSAUP must be verified. To do this, we formalise in Smirnov [22] the concept of the structural stability threshold of the model.

Definition 3. If the original model satisfies condition RNDSAUP and the price prehistory is known, structural stability threshold $\mathfrak{p}_t(K_t(\cdot))$ of the model at time t equals $+\infty$ if condition RNDSAUP is satisfied for each perturbation $\tilde{K}_t(\cdot)$ of the model; otherwise, it is defined by two conditions:

a) condition RNDSAUP is satisfied for each perturbed model satisfying inequality $d_{PH}(\mathrm{conv}(K_t(\cdot)), \mathrm{conv}(\tilde{K}_t(\cdot))) < \mathfrak{p}_t(K_t(\cdot))$;

b) there exists a perturbed model such that $d_{PH}(\mathrm{conv}(K_t(\cdot)), \mathrm{conv}(\tilde{K}_t(\cdot))) > \mathfrak{p}_t(K_t(\cdot))$ and condition RNDSAUP is not satisfied.

An explicit expression for the structural stability threshold and its properties are obtained in Smirnov [22]. These results are helpful to estimate the sensitivity of the solutions of (1) for an initial model, satisfying the conditions of Theorem 1, with respect to uniformly small perturbations of compacta $K_t(\cdot)$. It is to stress that we do not need any kind of "smoothness" conditions (like

[19]The assumption of *uniform* continuity of multivalued mappings in Proposition 3.7 of Carassus, Obłój and Wiesel [21] is redundant. In fact, it is used only for the arguments from the set of possible trajectories B_{t-1}, which is compact.

[20]In the case considered here, it is equivalent to NDAO condition, in our terminology.

[21]This is a verbal expression from Carassus, Obłój and Wiesel [21]; a more correct way to formulate it would be to say that the functions $v_t(\cdot)$ admit a continuous version.

semicontinuity or continuity) or even measurability for the compact-valued mapping $\tilde{K}_t(\cdot)$, describing the price dynamics of perturbed model.

For non-void compacta K define function r by

$$r(K) = \min_{h \in S_1(0)} \sigma_K(h), \tag{3}$$

where σ_K stands for the support function of K. If $0 \in \mathrm{int}(\mathrm{conv}(K))$, the quantity $r(K)$ has a nice geometric interpretation as the (positive) distance from the point 0 to the boundary of the convex hull of the set K. It turns out that the structural stability threshold in the case of no trading constraints is just $r(K_t(\cdot))$ and if the multifunction $K_t(\cdot)$ is continuous then[22]

$$r_t^* = \inf_{x \in B_t} r(K_t(x)) > 0, \tag{4}$$

which can be interpreted as guaranteed (worst) structural stability threshold. Note that the continuity modulus of functions v_s^* in Theorem 1 depends on r_t^*, defined by (4).

In the approximation result from Smirnov [23], presented below, we suppose that the trading constraints $D_t(\cdot)$ are representable in the form of Motzkin decomposition, introduced in Goberna, González, Martínez-Legaz and Todorov [24].

Theorem 3. Suppose that for initial model payoff functions $g_t(\cdot)$ and compact-valued mappings are $K_t(\cdot)$ are continuous, the multifunctions $D_t(\cdot)$ are closed and lower semicontinuous[23], and $D_t(\cdot)$ is Motzkin decomposable, i. e. can be represented as Minkowski sum

$$D_t(\cdot) = D_t^1(\cdot) + D_t^2(\cdot), \quad t \in \{1, \ldots, N\}, \tag{5}$$

where $D_t^1(\cdot) = \mathrm{rec}(D_t(\cdot))$ is recession cone of $D_t(\cdot)$ and $D_t^2(\cdot)$ is compact; additionally, we assume that $D_t^2(\cdot)$ can be chosen such that the multifunctions $x \mapsto D_t^2(x)$ are continuous; suppose also that[24]

$$\inf_{x \in B_{t-1}} \mathfrak{p}_t(K_t(x)) = \mathfrak{p}_t^* > 0 \tag{6}$$

for $t = 1, \ldots, N$. Then for the uniformly close perturbations

$$\sup_{x \in B_t} d_{PH}(K_t(x), \tilde{K}_t(x)) \le \delta$$

there is a constructive estimate for the uniform approximation of the solution of Bellman–Isaacs equations (1)

$$\sup_{x \in B_t} |v_t^*(x) - \tilde{v}_t^*(x)| \le \psi(\delta) \to 0 \quad \text{as } \delta \to 0. \tag{7}$$

[22] In fact, under this assumption the greatest lower bound below is attained.

[23] These assumptions about $g_t(\cdot)$, $K_t(\cdot)$ and $D_t(\cdot)$, together with RNDSAUP (which follows from the assumption (6) formulated below) imply the continuity of the Bellman–Isaacs equations according Theorem 1.

[24] A sufficient condition for this inequality is given in Smirnov [22].

We show that if structural stability does not hold, continuous superhedging pricing can be "fragile": it means that uniformly small (with respect to d_{PH} metric) perturbations of the model's dynamics can dramatically change superhedging prices; it is evident when a "no arbitrage" condition is broken, but it can be the case even if the "no arbitrage" condition is preserved. Below we formulate a result (not yet published) concerning the "fragility" of superhedging prices for one-step (i.e. $N = 1$) deterministic price dynamics model, when the condition NDAO holds, but RNDAO is not satisfied.

Theorem 4. Assume that that there are no trading constraints, the number of risky assets $n \geq 2$, the initial deterministic model is defined by a continuous multifunction $K_1(\cdot)$ with (non-void) compact convex values such that NDAO holds and the set[25] $B_0^* = \{x \in B_0 : \; n-1 \geq \dim(\operatorname{span}(K_1(\cdot))) > 0\}$ is non-void[26]. Then there exist a continuous payoff function g_1, a number $\beta > 0$ and a non-void Borel subset B_0^{**} of B_0^*, such that for every $\delta > 0$ there exist a perturbed model satisfying NDAO, defined by a Borel-measurable multifunction $\tilde{K}_1(\cdot) = \tilde{K}_1^{(\delta)}(\cdot)$ with (non-void) compact convex values from, uniformly close to initial model in the Pompeiu–Hausdorff metric: $\sup_x d_{PH}(\tilde{K}_1(x), K_1(x)) < \delta$, whereas the solutions of (1), v_t^* and $\tilde{v}_t^*$ for initial and perturbed models respectively, differs: $v_t^*(x) - \tilde{v}_t^*(x) \geq \beta$ for all $x \in B_0^{**}$.

Realistic models and structural stability. We regard as "realistic" a stochastic model of market behaviour (a stochastic process with discrete time describing price evolution) if the conditional distributions of the current price depend "continuously" on the price history. This is a natural property of market models from an economic point of view: there are no economic grounds for discontinuity of the dependence on price prehistory. The following is a formalisation of this property.

Definition 4. We say that a stochastic model of price evolution is realistic, if the transition kernels Q_t corresponding to the conditional probabilities of the price $X_t \in \mathbb{R}^n$ at time t with a known history $\bar{X}_{t-1} = \bar{x}_{t-1} \in (\mathbb{R}^n)^t$ admit a version satisfying the Feller property, i.e. the mapping $x \mapsto P_t(x, \cdot)$ is continuous[27] in the weak topology on the space of probability measures. Note that the deterministic and stochastic approaches lead to the same notions of "no arbitrage" in terms of $\operatorname{conv}(\operatorname{supp}(P_t(x, \cdot)))$, when the reference probability measure is specified by means of the Ionescu Tulcea construction using Feller transition kernels.

In the context of the deterministic approach, we propose the following formalisation of a "realistic" model of price evolution.

Definition 5. We call a deterministic model realistic if there exist mixed

[25] Here $\operatorname{span}(A)$ stands for linear span (linear hull) of a set $A \subseteq \mathbb{R}^n$.
[26] Hence, RNDAO is not satisfied.
[27] For measurability issues, see Proposition 1 in Smirnov [25].

market strategies $P_t(x,\cdot)$ representable as Feller transition kernels,[28] satisfying the consistency condition (2).

A necessary and sufficient condition for the existence of such a selector P_t is lower semicontinuity of the multifunctions[29] F_t, as shown in Smirnov [25], Theorem 2.

In what follows, we assume that there are no trading constraints, i.e. $D_t(\cdot) \equiv \mathbb{R}^n$. It turns out that under the assumption of structural stability, i.e. the RNDAO condition, and some stronger assumptions about the multifunctions $K_t(\cdot)$, the assertion of Smirnov [25], Theorem 2, concerning the existence of a Feller kernel selector, can be strengthened; moreover, it can be shown in a constructive manner.

Theorem 5. If the set of initial prices K_0 is convex and compact, the values of the continuous multifunctions $K_t(\cdot)$ are convex compact sets, and RNDAO holds, then there exists a stochastic model, satisfying the consistency condition (2), such that the kernels P_t are strong Feller in the strict sense.[30]

It is interesting, that the module of continuity for constructed Feller kernel (considering dependence of the probabilities on price prehistory) depend on depends on the guaranteed (worst) structural stability threshold, given by (4), see Smirnov [27].

The structural stability for a probabilistic model. As the 2021 session of the International Seminar on Stability Problems for Stochastic Models commemorates the 90th birthday of the outstanding mathematician Vladimir Zolotarev (27.02.193–07.11.2019), founder of this seminar, we would like to take the opportunity to mention Zolotarev's considerable contribution to the theory of probability metrics, see e. g. Zolotarev [28]. We show the *preservation of structural stability for transitional kernel perturbations, small enough with respect to one of the three probability metrics considered.* This is one more argument in favour of the importance of structural stability for financial modelling.

For a monotone nondecreasing function $f : [0, \infty) \to \mathbb{R}$, we define the upper generalised inverse by

$$f^{\lfloor -1 \rfloor}(y) = \inf\{x \in [0, \infty) : f(x) > y\} = \sup\{x \in [0, \infty) : f(x) \le y\},$$

which is nondecreasing right-continuous. Denote the open half-space with normal vector u by $H_u = \{y \in \mathbb{R}^n : \langle u, y\rangle > 0\}$, the ϵ-neighbourhood of the set B by $B^\epsilon = \{z \in \mathbb{R}^n : \rho(z, B) < \epsilon\}$, where ρ is Euclidian metric on $\mathbb{R}^n$ and

[28] This can be interpreted as a smooth version of conditional distributions X_t given prehistory $\bar{X}_{t-1} = x$.

[29] Or, equivalently, lower semicontinuity of the multifunctions K_t.

[30] We follow the terminology of Revuz [26] Chapter 1, Definition 5.8. The kernels P_t are iff the mapping $x \to P_t(x, \cdot)$ is continuous in the metric on the space of probability measures (equipped with a σ-algebra), generated by the total variation norm on the space of finite alternating measures.

$\rho(z, B) = \inf_{x \in B} \rho(z, x)$. Consider three metrics[31] on the space of probability measures on[32] $(\mathbb{R}^n, \mathcal{B}^n)$:

$$d_{UC}(Q', Q) = \sup_{A \in \mathcal{C}_{\mathbb{R}^n}} |Q'(A) - Q(A)|,$$

where $\mathcal{C}_{\mathbb{R}^n}$ is the class of all non-void convex subsets of $\mathbb{R}^n$;

$$l_\infty(Q', Q) = \inf\{\epsilon > 0 : Q(B) \le Q'(B^\epsilon),\ Q'(B) \le Q(B^\epsilon) \text{ for all } B \in \mathcal{B}^n\};$$

and the Prokhorov metric

$$d_P(Q', Q) = \inf\{\epsilon > 0 : Q(B) \le Q'(B^\epsilon) + \epsilon,\ Q'(B) \le Q(B^\epsilon) + \epsilon \text{ for all } B \in \mathcal{B}^n\}.$$

The following result is proved in Smirnov [30].

Theorem 6. Let Q_t be the kernels of the initial model and $\tilde{Q}_t$ those of the perturbed model. Assume that there are no trading constraints and that for the initial model RNDAO holds (using the consistency condition (2)).

1. Then[33] the perturbed model with kernels $\tilde{Q}_t$ satisfies RNDAO if it is close to the initial one in the sense that

   $$d_{UC}(\tilde{Q}_t(x, \cdot), Q_t(x, \cdot)) < p_t^*(x),$$

   where $p_t^*(x) = \inf_{u \in S_1(0)} Q_t(x, H_u) > 0$.

2. Suppose that the supports of $\tilde{Q}_t(x, \cdot)$ and $Q_t(x, \cdot)$ are compact. Then the perturbed model with kernels $\tilde{Q}_t$ satisfies RNDAO if it is close to the initial one in the sense that[34]

   $$l_\infty(\tilde{Q}_t(x, \cdot), Q_t(x, \cdot)) < r_t^*(x),$$

   where $r_t^*(x) = r(K_t(x)) > 0$ and r is given by (3).

3. There exists a $d_t^*(x) \in (0, 1)$ that can be defined, given the measure $Q_t(x, \cdot)$, such that the perturbed model with kernels $\tilde{Q}_t$ satisfies RNDAO if[35]

   $$d_P(\tilde{Q}_t(x, \cdot), Q_t(x, \cdot)) < d_t^*(x).$$

[31] Note that l_∞ is the minimal metric with respect to the metric on the space of random vectors defined by $L_\infty(X', X) = \operatorname{vrai\,max} \rho(X', X)$, see, e. g., (7.5.15) in Rachev, Klebanov, Stoyanov and Fabozzi [29]. The metric d_{UC} is a generalisation of the Kolmogorov metric d_K defined for probabilities on the real line $\mathbb{R}$. On $\mathbb{R}$, these two metrics are equivalent: $d_K \le d_{UC} \le 2d_K$. Note that the metric d_{UC} is used, for example, to estimate the speed of convergence in the multidimensional central limit theorem, see Bentkus [30]. The Prokhorov metric d_P metrises the weak topology on the space of probabilities on a separable metric space, see Prokhorov [31].

[32] Here $\mathcal{B}^n$ stands for the Borel σ-algebra.

[33] In this paragraph, the supports $Q_t(x, \cdot)$ and $\tilde{Q}_t(x, \cdot)$ are not necessarily compact and the result generalise that of Ostrovski [5]

[34] The RNDAO condition (for the initial model) is tantamount to $r(K_t(x)) > 0$ for all x and t.

[35] In this case also, the supports $Q_t(x, \cdot)$ and $\tilde{Q}_t(x, \cdot)$ are not necessarily compact.

If, additionally, the supports of $Q_t(x, \cdot)$ are compact, then it is sufficient to set

$$d_t^*(x) = \phi_{t,x}^{\lfloor -1 \rfloor}(r(K_t(x))),$$

where r is given by (3),

$$\phi_{t,x}(z) = z + \psi_{t,x}^{\lfloor -1 \rfloor}(z),$$

and

$$\psi_{t,x}(u) = \inf_{y \in K_t(x)} Q(x, \bar{B}_u(y)).$$

References

1. A. A. Andronov, L. S. Pontryagin, Systèmes grossiers *Dokl. Akad. Nauk SSSR* **14**:5 (1937) 247–250.

2. W. Schachermayer, The fundamental theorem of asset pricing, *Mathematical Finance*, **14** (2004) 19–48.

3. E. Bayraktar, Y. Zhang, Z. Zhou., A note on the fundamental theorem of asset pricing under model uncertainty, *Risks* **2** (2014) 425–433.

4. Z. Hou, J. Obłój, Robust pricing–hedging dualities in continuous time, *Finance Stoch*, **22** (2018), 511–567.

5. V. Ostrovski, Stability of no-arbitrage property under model uncertainty, *Statistics and Probability Letters*, **83** (2013) 89–92.

6. S. N. Smirnov, A Guaranteed Deterministic Approach to Superhedging: Financial Market Model, Trading Constraints, and the Bellman–Isaacs Equations, *Automation and Remote Control* **82**:4 (2021) 722–743.

7. T. Matsuda, A. Takemura, Game-theoretic derivation of upper hedging prices of multivariate contingent claims and submodularity, *Jpn J. Ind. Appl. Math.* **37** (2020) 213–248.

8. S. N. Smirnov, A guaranteed deterministic approach to superhedging: Sensitivity of solutions of Bellman–Isaacs equations and numerical methods, *Computational Mathematics and Modeling* **31** (2020) 384–401.

9. S. N. Smirnov, General theorem on a finite support of mixed strategy in the theory of zero-sum games, *Doklady Mathematics* **97** (2018) 215–218.

10. B. Bouchard and M. Nutz, Arbitrage and duality in nondominated discrete-time models, *Annals of Applied Probability* **25** (2015) 823–859.

11. M. Burzoni, M. Frittelli, Z. Hou, M. Maggis, and J. Obłój, Pointwise arbitrage pricing theory in discrete time, *Math. Oper. Res.* **44** (2019) 1034–1057.

12. L. Carassus and T. Vargiolu, Super-replication price: It can be OK, *ESAIM: Proceedings and Surveys* **65** (2018) 241–281.

13. V. N. Kolokoltsov, Nonexpansive maps and option pricing theory, *Kybernetika* **34** (1998) 713–724.

14. P. Bernhard, J. C. Engwerda, B. Roorda, J. Schumacher, V. Kolokoltsov, P. Saint-Pierre, and J.-P. Aubin, *The Interval Market Model in Mathematical Finance: Game-Theoretic Methods*, Springer, New York, 2013.

15. J. Obłój and J. Wiesel, A unified framework for robust modelling of financial markets in discrete time, arXiv:1808.06430v2 (2019).

16. S. N. Smirnov, A Guaranteed Deterministic Approach to Superhedging: No Arbitrage Properties of the Market, *Automation and Remote Control* **82**:1 (2021) 172–187.

17. S. N. Smirnov, Geometric criterion for a robust condition of no sure arbitrage with unlimited profit, *Moscow University Computational Mathematics and Cybernetics* **44**:3 (2020) 146–150.

18. J. Jacod and A. N. Shiryaev, Local martingales and the fundamental asset pricing theorems in the discrete-time case, *Finance And Stochastics*, **2** (1998) 259–273.

19. S. N. Smirnov, The guaranteed deterministic approach to superhedging: Lipschitz properties of solutions of the Bellman–Isaacs equations, in *Frontiers of Dynamics Games: Game Theory and Management* Birkhäuser, St. Petersburg, 2019, 267–288.

20. S. N. Smirnov, A guaranteed deterministic approach to superhedging: Relation between “deterministic” and “stochastic” settings in the case of no trading constraints, *Theory Probab. Appl.* [to appear].

21. L. Carassus, J. Obłój, and J. Wiesel, The robust superreplication problem: A dynamic approach, *SIAM Journal on Financial Mathematics* **10** (2019) 907–941.

22. S. N. Smirnov, Structural Stability Threshold for the Condition of Robust No Deterministic Sure Arbitrage with Unbounded Profit, *Moscow University Computational Mathematics and Cybernetics* **45**:1 (2021) 34–44.

23. S. N. Smirnov, A guaranteed deterministic approach to superhedging: Structural stability and approximation, *Computational Mathematics and Modeling* [to appear]

24. M. A. Goberna, E. González, J. E. Martínez-Legaz and M. I. Todorov, Motzkin decomposition of closed convex sets, *Journal of Mathematical Analysis and Applications* **364**:1 (2010) 209–221.

25. S. N. Smirnov, A Feller transition kernel with measure supports given by a set-valued mapping, *Proceedings of the Steklov Institute of Mathematics* **308** (2020) S188–S195.

26. D. Revuz, *Markov chains*, North Holland, Amsterdam, 1975.

27. S. N. Smirnov, Realistic models of financial market and structural stability, *Journal of Mathematics* [to appear]

28. V. M. Zolotarev, Probability metrics, *Theory of Probability and its Applications* **28**:2 (1983) 264–287.

29. S. Rachev, L. Klebanov, S. Stoyanov and F. Fabozzi, *The Methods of Distances in the Theory of Probability and Statistics*, Springer-Verlag, New York, 2013.

30. R. Bentkus, On the dependence of the Berry–Esseen bound on dimension, *Journal of Statistical Planning and Inference* **113** (2003) 385–402.

31. Y. V. Prokhorov, Convergence of random processes and limit theorems in probability theory, *Theory Probab. Appl.* **1** (1956) 157–214.