# Structural Stability of Financial Market Models

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# Introduction

# Aim of the Presentation

The main aim of this presentation is to demonstrate the importance of structural stability for financial modelling, in particular, discuss its relation to the continuity and approximation properties of superhedging prices.

**The structural stability is the fundamental property of a model, which means that the qualitative behaviour of the model is unaffected by small (in a certain sense) perturbations of its dynamics.** The term "structural stability" is borrowed from dynamical systems theory. Structurally stable systems were introduced in 1937 by Andronov and Pontryagin [1] under the name "systèmes grossiers," or coarse systems.

**From an economic point of view, such a qualitative behaviour of the model of the financial market is to admit no "arbitrage", in some sense to be made precise.**

# Literature on Stability of No-Arbitrage Property

The idea of stability of no arbitrage property under some perturbation of the model is, of course, present in the literature. We restrict ourselves to mention only few related papers.


1. W. Schachermayer, The fundamental theorem of asset pricing, *Mathematical Finance*, **14** (2004) 19–48.
2. E. Bayraktar, Y. Zhang, Z. Zhou., A note on the fundamental theorem of asset pricing under model uncertainty, *Risks* **2** (2014) 425–433.
3. Z. Hou, J. Obłój, Robust pricing–hedging dualities in continuous time, *Finance Stoch*, **22** (2018), 511–567.
4. V. Ostrovski, Stability of no-arbitrage property under model uncertainty, *Statistics and Probability Letters*, **83** (2013) 89–92.


**Depending on the context, the structural stability can be formalised in different ways**.

# Deterministic Market Model: a Formalisation of Uncertainty

- We propose a formalisation of structural stability in the framework of a *Guaranteed Deterministic Approach* (GDA), developed in a series of papers by the author. The main premise of GDA is based on a specific assumption concerning a priori information regarding price movements.
- The corresponding **market model formalise the "uncertainty" of price dynamics with discrete time and can be considered as purely deterministic**: the initial problem setting in the GDA framework does not use any probability or family of probabilities, see

S. N. Smirnov, A Guaranteed Deterministic Approach to Superhedging: Financial Market Model, Trading Constraints, and the Bellman–Isaacs Equations, *Automation and Remote Control* **82**:4 (2021) 722–743.

# Guaranteed Deterministic Approach to Superhedging Problem

- Concerning the superhedging problem within the GDA framework, we should stress that we adopt an alternative interpretation to the common robust approach to pricing of contingent liabilities. Our **interpretation is game-theoretic**: we deal with a deterministic dynamic two-player zero-sum game of "hedger" against "market."
- A family of probabilities appears as a secondary notion, thanks to the introduction of **mixed strategies of the "market"**; this makes it possible to use the mathematical techniques based on the game equilibrium instead of the usual duality method.

# Relation to Robust Modelling Approaches

Formally, from the contemporary point of view, the guaranteed deterministic approach to the superhedging problem can be classified as

- a specific pathwise (or pointwise) approach addressing uncertainty in market modelling by defining a set of deterministic market scenarios, a result of agents beliefs,
- or it can be formally described in terms "quasi-sure" approach by the choice of a collection of probabilistic models (possible priors) for the market.

For these two robust modelling approaches and for detailed review of large literature focusing on robust approach to mathematical finance we refer to

- B. Bouchard and M. Nutz, Arbitrage and duality in nondominated discrete-time models, *Annals of Applied Probability* **25** (2015) 823–859.
- M. Burzoni, M. Frittelli, Z. Hou, M. Maggis, and J. Obłój, Pointwise arbitrage pricing theory in discrete time, *Math. Oper. Res.* **44** (2019) 1034–1057.

# Kolokoltsov's Contribution

To the best of our knowledge, the first work to explicitly articulate GDA framework to pricing and hedging contingent claims was[1] the paper

- V. N. Kolokoltsov, Nonexpansive maps and option pricing theory, *Kybernetika* **34** (1998) 713–724.

The GDA is closely related to a class of market models called interval models in

- P. Bernhard, J. C. Engwerda, B. Roorda, J. Schumacher, V. Kolokoltsov, P. Saint-Pierre, and J.-P. Aubin, *The Interval Market Model in Mathematical Finance: Game-Theoretic Methods*, Springer, New York, 2013,

especially to the results of Kolokoltsov in Chapters 11–14, including independent discovery of the **game-theoretic interpretation of risk-neutral probabilities under the assumption of no trading constraints.**

[1] The guaranteed deterministic approach was developed by us in late 90-s (although at that period we were not aware of Kolokoltsov's paper), but published only in the last three years, together with some recent new results.

# Deterministic Financial Market Model

# Financial Market Model

Consider discounted prices of $n$ risky assets; without loss of generality, we can suppose that the risk-free asset has a fixed price, equal to one, and so in what follows we call "discounted prices" simply "prices."

Let $X_t$ be the price vector at time $t$ and $\Delta X_t = X_t - X_{t-1}$ be the vector of price increments. The above-mentioned assumption about a priori information describing the uncertainty of price movements is formalised as follows: **the price increments $\Delta X_t$ for $t = 1, \ldots, N$ lie in a priori given (non-void) closed sets**[2] **$K_t(\cdot) \subseteq \mathbb{R}^n$, depending on price prehistory.** The multivalued mappings $x \mapsto K_t(x)$ are assumed to be defined for all $x \in (\mathbb{R}^n)^t$, $t = 1, \ldots, N$. Denote by $B_t$ the set of possible trajectories (or paths) of asset prices in the time interval $[0, t]$, i.e.

$$B_t = \{(x_0, \ldots, x_t) : x_0 \in K_0, \Delta x_1 \in K_1(x_0), \ldots, \Delta x_t \in K_t(x_0, \ldots, x_{t-1})\}.$$

[2]The dot "$\cdot$" indicates the variables representing the price evolution. More precisely, it indicates the prehistory $\bar{x}_{t-1} = (x_0, \ldots, x_{t-1}) \in (\mathbb{R}^n)^t$ for $K_t$, and it indicates the history $\bar{x}_t = (x_0, \ldots, x_t) \in (\mathbb{R}^n)^{t+1}$ for the functions $v_t^*$ and $g_t$ introduced below.

# Trading Constraints

We suppose that there are trading constraints, concerning only risky assets, described by a priori given sets $D_t(\cdot) \subseteq \mathbb{R}^n,\ t = 1, \ldots, N$, depending on price prehistory, which are assumed to be

- convex (hedger can diversify the portfolio),
- and such that $0 \in D_t(\cdot)$ (hedger can hold all the amount in the risk-free asset) .

The multivalued mappings $x \mapsto D_t(x)$ are assumed to be defined for all $x \in (\mathbb{R}^n)^t,\ t = 1, \ldots, N$. An admissible hedging strategy at time step $t$ is therefore $h \in D_t(\cdot)$.

A special case, often used in applications, is "no trading constraints" condition, which means that $D_t(\cdot) \equiv \mathbb{R}^n$.

# Options

Let us suppose that the "hedger" has a contingent liability on an option, defined by the given payoff functions $g_t(\cdot)$, describing the potential payouts on the option at time moment $t$.

- The functions $x \mapsto g_t(x)$, are assumed to be defined for all $x \in (\mathbb{R}^n)^t$, $t = 1, \ldots, N$.
- The functions $g_t(\cdot)$ are assumed to be bounded on $B_t$, the set of possible trajectories of asset prices in the time interval $[0, t]$
- In the case of an American option $g_t \geq 0$ for $t = 1, \ldots, N$ .
- European or Bermudian options can be also considered: if no payment is due at a moment of time $t$, we formally set $g_t \equiv -\infty$.
- It is convenient to assume (formally) that $g_0 \equiv -\infty$ (there are no liabilities to pay at the initial time).

# Guaranteed Deterministic Approach: Superhedging Problem

# Superhedging Problem

The superhedging is such a kind of hedging of contingent liability on an option which **covers completely this liability**, at the same time minimising the necessary initial endowment, i. e. premium.

To apply the GDA to the described above financial market model with trading constraints, we set the superhedging problem of a contingent claim on the American option, based on dynamic programming principle.

Let us denote by $v_t^*(\cdot)$ the infimum of the portfolio value at time $t$ that guarantees, given the price history, a choice of an appropriate hedging strategy covering current and future liabilities due to possible payments on the American option.

# Bellman–Isaacs equations

The corresponding Bellman–Isaacs equations **can be derived directly based on the economic sense**, by choosing at step $t$ the "best" admissible hedging strategy $h \in D_t(\cdot)$ for the "worst" scenario of (discounted) price increments $y \in K_t(\cdot)$ for the given payoff functions $g_t(\cdot)$, describing the potential payouts on the option; for $t = N, \ldots, 1$

$$v^*_{t-1}(\bar{x}_{t-1}) = g_{t-1}(\bar{x}_{t-1}) \vee \inf_{h \in D_t(\bar{x}_{t-1})} \sup_{y \in K_t(\bar{x}_{t-1})} \left[ v^*_t(\bar{x}_{t-1}, x_{t-1} + y) - hy \right],$$
$$v^*_N(\bar{x}_N) = g_N(\bar{x}_N), \tag{1}$$

where $\bar{x}_{t-1} = (x_0, \ldots, x_{t-1})$ represents the price history up to the present moment $t-1$, the symbol $\vee$ denotes maximum, and $hy = \langle h, y \rangle$ is the dot product of the vectors $h$ and $y$. In (1), the functions $v^*_t$ and the corresponding suprema and infima take values in an extended set of real numbers $\mathbb{R} \cup \{-\infty, +\infty\} = [-\infty, +\infty]$, which is the two-point compactification of $\mathbb{R}$.

# Relevant "No Arbitrage" Notions

# Relevant "No Arbitrage" Notions: Uncertainty Principle

The different notions of "no arbitrage" in the framework of the deterministic market model can be relevant .

As within DSA framework we formalise **uncertainty of price movement**, so a reasonable model should satisfy the following "uncertainty principle"[3]. If there is no "arbitrage" (in a certain sense) for the initial model with price dynamics described by $K_t(\cdot)$, then for the model with price dynamics described by $\tilde{K}_t(\cdot) \supseteq K_t(\cdot)$ and the same trading constraints the corresponding "no arbitrage" condition should be also valid. The meaning is rather clear: if we cannot realise "arbitrage" for initial model, then all the more we cannot realise "arbitrage" for a model with more uncertainty, i. e. with less information about price movement.

[3]Which is not the case for the traditional model with a reference probability, where the universe containing all states of the world is fixed (up to a set of zero measure) by means of consideration of equivalent measures only.

# Different Arbitrage Definitions

## Definition

- By deterministic arbitrage opportunity (DAO) at time step $t$, we mean that there exists a strategy $h^* \in D_t(\cdot)$ such that $h^* y \geq 0$ for all $y \in K_t(\cdot)$ and there exists a price movement $y^* \in K_t(\cdot)$ such that $h^* y^* > 0$.
- By deterministic sure arbitrage (DSA) at time step $t$, we mean that there exists a strategy $h^* \in D_t(\cdot)$ such that $h^* y > 0$ for all $y \in K_t(\cdot)$.
- We say that there is deterministic sure arbitrage with unlimited profit (DSAUP) at time step $t$ if the function $h \mapsto \inf\{hy,\ y \in K_t(\cdot)\}$ takes arbitrarily large values for $h \in D_t(\cdot)$.

# Related "No Arbitrage" Properties

Using these three notions of "arbitrage", we can define the corresponding "no arbitrage" properties on a time interval: each "no arbitrage" property on a time interval is tantamount to the corresponding "no arbitrage" property at every time step of this interval for any price prehistory.
So, we consider the following "no arbitrage" properties:

- no deterministic arbitrage opportunity (NDAO),
- no deterministic sure arbitrage (NDSA),
- no deterministic sure arbitrage with unbounded profit (NDSAUP)[4].

Note that **NDSA and NDSAUP satisfy the uncertainty principle**, whereas **NDAO, in general, do not satisfy this principle** and therefore can be considered as not relevant for the GDA framework (however, NDAO can satisfy this principle if certain additional assumptions are admitted, namely "robustness" property, defined below).

[4]Note that in the case of conic trading constraints (in particular, in the case of no trading constraints, i.e. $D_t(\cdot) \equiv \mathbb{R}^n$) NDSAUP coincide with NDSA.

# Structural stability: Robust "No Arbitrage" Notions

# Assumption about Boundedness of Price Increments

We share an idea, suggested in unpublished work of L. Carassus and T. Varioglu about 15 years ago and finally published in

L. Carassus and T. Vargiolu, Super-replication price: It can be OK, *ESAIM: Proceedings and Surveys* **65** (2018) 241–281.

**In order to get a meaningful theory, it is reasonable to assume the boundedness of price increments**. This reflects a certain quality of the information available about price behaviour; if this assumption is not valid, the typical superhedging strategy would be "buy and hold", i. e. trivial one.

Hereinafter, we assume that all the sets $K_t(\cdot)$ are bounded (hence compact), unless otherwise stated.

# Robust "No Arbitrage" Properties

Since our interpretation appeals to a vague knowledge about the price behaviour, we have introduced an important concept of structural stability. In our context it is formalised as follows: **a specific "no arbitrage" property (which reflects the qualitative behaviour of a price dynamics) is unaffected by perturbations of $K_t(\cdot)$ that are sufficiently small with respect to the Pompeiu–Hausdorff metric** $d_{PH}$; we call such a "no arbitrage" property robust[5] or coarse.

In what follows, we consider two coarse "no arbitrage" properties: robust no deterministic arbitrage opportunity (RNDAO) and robust no deterministic sure arbitrage with unlimited profit (RNDSAUP). More precisely it is defined below.

[5]We feel now that the term "robust" is overused in the literature (with different meanings) and the term "coarse" would be better, but unfortunately "robust" is already used in our papers.

# Definition of "Robust No Arbitrage" Properties

## Definition

Suppose that the initial market model satisfies the NDAO (respectively NDSAUP) condition and consider a perturbed model with increments $\Delta X_t$ lying in compact sets $\tilde{K}_t(\cdot)$ for $t = 1, \ldots, N$.

The robust (or coarse) NDAO (respectively NDSAUP) property, abbreviated RNDAO (respectively RNDSAUP), means that for any $t \in \{1, \ldots, N\}$ and any prehistory of the prices $x \in B_{t-1}$, there exists $\epsilon_t(x) > 0$ such that if $d_{PH}(\tilde{K}_t(x), K_t(x)) \leq \epsilon_t(x)$, then the perturbed market model still satisfies the NDAO (respectively NDSAUP) condition

In fact, it can be weaken as follows: **the convex hulls of $\tilde{K}_t(x)$ and $K_t(x)$ need to be close** (with respect to the Pompeiu–Hausdorff metric), i.e. $d_{PH}(\mathrm{conv}(\tilde{K}_t(x)), \mathrm{conv}(K_t(x))) \leq \epsilon_t(x)$, where $\mathrm{conv}(A)$ is the convex hull of a set $A$. Note that $d_{PH}(\mathrm{conv}(A), \mathrm{conv}(B)) \leq d_{PH}(A, B)$ for compact sets $A$ and $B$.

# Geometric Criteria of "No Arbitrage" Properties

The corresponding geometric criteria were obtained in

- S. N. Smirnov, A Guaranteed Deterministic Approach to Superhedging: No Arbitrage Properties of the Market, *Automation and Remote Control* **82**:1 (2021) 172–187.
- S. N. Smirnov, Geometric criterion for a robust condition of no sure arbitrage with unlimited profit, *Moscow University Computational Mathematics and Cybernetics* **44**:3 (2020) 146–150.

Here we mention only two of them, which are of interest in the simple but important for applications **case of no trading constraints**. The condition NDAO in this case is equivalent to the Jacod–Shiryaev geometric criterion — in the probabilistic setting, this geometric criterion (understood almost surely) was found in

- J. Jacod and A. N. Shiryaev, Local martingales and the fundamental asset pricing theorems in the discrete-time case, *Finance And Stochastics*, **2** (1998) 259–273.

and RNDAO is equivalent to the "enhanced" Jacod–Shiryaev geometric criterion; this criteria are presented on the next slide.

# Geometric Criteria of Jacod–Shiryaev type

## Jacod–Shiryaev geometric criterion of NDAO

$$0 \in \mathrm{ri}(\mathrm{conv}(K_t(\cdot))),\ t = 1, \ldots, N,$$

where $\mathrm{ri}(A)$ is the relative interior of a convex set $A$.

## "Enhanced" Jacod–Shiryaev geometric criterion of RNDAO

$$0 \in \mathrm{int}(\mathrm{conv}(K_t(\cdot))),\ t = 1, \ldots, N.$$

- Note that in general RNDSAUP does not imply $\mathrm{int}(\mathrm{conv}(K_t(\cdot))) \neq \emptyset$.
- In the case of no trading constraints RNDSAUP is equivalent to RNDAO.

# Continuity of Pricing and Structural Stability

# Continuity of Solutions of Bellman–Isaacs Equations

## Theorem (on Continuity)

*Suppose that*

- *RNDSAUP holds,*
- *for $s = 1, \ldots, N$, the payoff functions $g_s$ are continuous*
- *compact-valued mappings $\bar{x}_{s-1} \mapsto K_s(\bar{x}_{s-1})$ are h-continuous (continuity with respect to the Pompeiu–Hausdorff metric[a]).*

*Then, the functions $v_s^*$, defined by (1), are uniformly continuous and bounded on $B_s$; moreover, the continuity modulus of functions $v_s^*$ can be estimated with the help of recurrent inequalities.*

[a] For compact-valued mappings, *h*-continuity is equivalent to continuity.

# Continuity Results

In the case of no trading constraints[6] this result was obtained in

- S. N. Smirnov, The guaranteed deterministic approach to superhedging: Lipschitz properties of solutions of the Bellman–Isaacs equations, in *Frontiers of Dynamics Games: Game Theory and Management* Birkhäuser, St. Petersburg, 2019, 267–288.

In general case the continuity result was obtained in

- S. N. Smirnov, A guaranteed deterministic approach to superhedging: Sensitivity of solutions of Bellman–Isaacs equations and numerical methods, *Computational Mathematics and Modeling* **31** (2020) 384–401.

[6] Recall that in this case RNDSAUP is equivalent to RNDAO.

# Related Probabilistic Model

Let us introduce a related probabilistic model, assuming that there are no trading constraints. Suppose that the initial price takes values in some compact set $F_0 = K_0$. Denote $F_t(x) = x + K_t(x),\ x \in B_{t-1},\ t = 1, \dots, N$. Consider a family of multifunctions $F_t(\cdot),\ t = 1, \dots, N$ and a family $\mathcal{P}_t$ of the stochastic kernels $P_t,\ t = 1, \dots, N$; the measure $P_t(\bar{x}_{t-1}, \cdot)$ is interpreted as a conditional distribution of $X_t$ given the price prehistory $\bar{X}_{t-1} = \bar{x}_{t-1} \in B_{t-1}$. A measure $P_0$ is interpreted as a (marginal) distribution of $X_0$. The corresponding probability $\mathbb{P}$ can be defined using Ionescu Tulcea construction.

We say that the **consistency condition** is satisfied if

**Relation between stochastic and deterministic models**

$$\operatorname{supp}(P_t(x, \cdot)) = F_t(x), \quad x \in B_{t-1},\ t = 1, \dots, N; \qquad \operatorname{supp}(P_0) = F_0.$$

# Superhedging: deterministic and stochastic settings

We prove the following results in

- S. N. Smirnov, A guaranteed deterministic approach to superhedging: Relation between "deterministic" and "stochastic" settings in the case of no trading constraints, *Theory Probab. Appl.* [to appear].

Under quite general assumptions GDA pricing is not less the probabilistic pricing (almost surely) if consistency condition holds and there are simple examples where it is strictly greater.

**Theorem 2**

In the case of no trading constraints, under the same assumptions as in Theorem 1, the superhedging prices for the American option in the usual probabilistic setting, satisfying the consistency condition, coincide almost surely with the corresponding solutions of (1).

As a consequence, under assumptions of Theorem 2 the superhedging prices for the usual probabilistic setting admit a continuous version.

# A Result of L. Carassus, J. Obłój, and J. Wiesel

We have noticed, that Proposition 3.7 in the Section 3.2 of the paper

- L. Carassus, J. Obłój, and J. Wiesel, The robust superreplication problem: A dynamic approach, *SIAM Journal on Financial Mathematics* **10** (2019) 907–941.

about the continuity (as a function of price history) of superhedging price is not valid unless one of its assumptions is strengthen. Using our terminology we can formulate the result of this proposition in equivalent form. The assumptions are the following: The initial price is assumed to be fixed, $X_0 = x_0$; the correspondences $F_t(\cdot)$ are compact-valued and uniformly[7] continuous; the payoff function $g$ on the European option is continuous; no quasi-sure arbitrage condition[8] with respect to the $\mathcal{P}_t$, the set of priors at time $t$, which consists of all the kernels, satisfying the consistency condition, holds for $t = 1, \ldots, N$.

[7] The assumption of *uniform* continuity is redundant. In fact, it is used only for the arguments from the set of possible trajectories $B_{t-1}$, which is compact.

[8] In the case considered here, it is equivalent to NDAO condition, in our terminology.

# Structural Stability is Essential for Continuity

The proposition in question states that under these assumptions the quasi-sure superhedging price $V_t,\ t = 0, \dots, N$ coincides with the almost sure superhedging price, and the functions $V_t = v_t(\bar{X}_t)$ are continuous[9] for $t = 1, \dots, N$.

We have constructed a counterexample clarifying the origin of the mistake in this assertion and showing, that **NDAO condition is still insufficient for the continuity of the superhedging price**.

Thus, the structural stability[10] is essential to this kind of continuity properties.

[9] This is a verbal expression from the paper by Carassus, Obłój and Wiesel; a more correct way to formulate it would be to say that the functions $v_t(\cdot)$ admit a continuous version.

[10] In this case RNDAO condition is suitabale (instead of NDAO).

# Dynamics Approximation and Structural Stability

# Dynamics Approximation and Structural Stability Threshold

For the original market model, a natural way of solving the problem approximately is to construct a perturbed market model such that compacta $\tilde{K}_t(\cdot)$, describing the uncertainty of price movement have simple structures (e.g., they could be finite sets).

To preserve the economic meaning of the solution to the problem for the perturbed market model (which is to have qualitative properties similar to those of the original system), we must preserve the structural stability conditions. If using the perturbed market model we obtain a numerical solution with the prescribed error and such that the price increments lie in the compacta $\tilde{K}_t(\cdot)$, the meeting of condition RNDSAUP must be verified. To control this, we formalise the concept of the structural stability threshold (SST) of the model in the paper

- S. N. Smirnov, Structural Stability Threshold for the Condition of Robust No Deterministic Sure Arbitrage with Unbounded Profit, *Moscow University Computational Mathematics and Cybernetics* **45**:1 (2021) 34–44.

# Definition of Structural Stability Threshold

## Definition

If the original model satisfies condition RNDSAUP and the price prehistory is known, structural stability threshold $\mathfrak{p}_t(K_t(\cdot))$ of the model at time t equals $+\infty$ if condition RNDSAUP is satisfied for each perturbation $\tilde{K}_t(\cdot)$ of the model; otherwise, it is defined by two conditions:

1. condition RNDSAUP is satisfied for each perturbed model satisfying inequality $d_{PH}(\mathrm{conv}(K_t(\cdot)), \mathrm{conv}(\tilde{K}_t(\cdot))) < \mathfrak{p}_t(K_t(\cdot))$;
2. there exists a perturbed model such that $d_{PH}(\mathrm{conv}(K_t(\cdot)), \mathrm{conv}(\tilde{K}_t(\cdot))) > \mathfrak{p}_t(K_t(\cdot))$ and condition RNDSAUP is not satisfied.

- An explicit expression for the structural stability threshold and its properties are obtained in the paper cited on the previous slide.
- It turns out that these results are helpful to estimate the sensitivity of the solutions of (1) for an initial model, satisfying the conditions of Theorem 1, with respect to uniformly small perturbations.

For non-void compacta $K$ define function $r$ by

$$r(K) = \min_{h \in S_1(0)} \sigma_K(h) = r(\mathrm{conv}(K)), \tag{2}$$

where $\sigma_K$ stands for the support function of $K$. If $0 \in \mathrm{int}(\mathrm{conv}(K))$, the quantity $r(K)$ has a nice geometric interpretation as the (positive) distance from the point 0 to the boundary of the convex hull of the set $K$.

- It is easy to see that **in the case of no trading constraints the structural stability threshold is just** $r(K_t(\cdot))$. Moreover, if the multifunction $K_t(\cdot)$ is continuous then

$$r_t^* = \inf_{x \in B_t} r(K_t(x)) = \min_{x \in B_t} r(K_t(x)) > 0, \tag{3}$$

  which can be considered as guaranteed (worst) structural stability threshold.
- Note that n the case of no trading constraints the continuity modulus of functions $v_s^*$ in Theorem 1 depends on $r_t^*$, defined by (3).

# Approximation and Sensitivity

It is to stress that for approximation results we do not need any kind of "smoothness" conditions (like semicontinuity or continuity) or even measurability for the compact-valued mapping $\tilde{K}_t(\cdot)$, describing the price dynamics of perturbed model.

In the approximation result from

- S. N. Smirnov, A guaranteed deterministic approach to superhedging: Structural stability and approximation, *Computational Mathematics and Modeling* [to appear]

presented below, we suppose that the trading constraints $D_t(\cdot)$ are in the form of Motzkin decomposition, introduced in

- M. A. Goberna, E. González, J. E. Martínez-Legaz and M. I. Todorov, Motzkin decomposition of closed convex sets, *Journal of Mathematical Analysis and Applications* **364**:1 (2010) 209–221.

# Approximation Theorem

## Theorem 3

For $t = 1, \ldots, N$ suppose that for initial model payoff functions $g_t(\cdot)$ and compact-valued mappings are $K_t(\cdot)$ are continuous, the multifunctions $D_t(\cdot)$ are closed and lower semicontinuous[a], and $D_t(\cdot)$ is Motzkin decomposable, i. e. can be represented as Minkowski sum

$$D_t(\cdot) = D_t^1(\cdot) + D_t^2(\cdot), \quad t \in \{1, \ldots, N\},$$

where $D_t^1(\cdot) = \mathrm{rec}(D_t(\cdot))$ is recession cone of $D_t(\cdot)$ and $D_t^2(\cdot)$ is compact; assume that $D_t^2(\cdot)$ can be chosen such that the multifunctions $x \mapsto D_t^2(x)$ are continuous; suppose also that

$$\inf_{x \in B_{t-1}} \mathfrak{p}_t(K_t(x)) = \mathfrak{p}_t^* > 0 \tag{4}$$

[a] These assumptions about $g_t(\cdot)$, $K_t(\cdot)$ and $D_t(\cdot)$, together with RNDSAUP, which follows from the assumption (4), imply the continuity of solutions of the Bellman–Isaacs equations, according Theorem 1.

# Approximation Theorem — Continuation

## Theorem 3 (continuation)

Then for the uniformly close perturbations of initial dymamics

$$\sup_{x \in B_t} d_{PH}(K_t(x), \tilde{K}_t(x)) \leq \delta$$

there is a constructive estimate for the uniform approximation of the solution of Bellman–Isaacs equations (1)

$$\sup_{x \in B_t} |v_t^*(x) - \tilde{v}_t^*(x)| \leq \psi(\delta) \to 0 \quad \text{as } \delta \to 0. \tag{5}$$

The constructive estimate for the uniform approximation of the solution of Bellman–Isaacs equations is helpful to estimate the precision of the numerical method, based on an approximation of the initial sets $K_t(x))$ by finite sets $\tilde{K}_t(x))$, chosen as subsets of a lattice[11] on $\mathbb{R}^n$.

[11] A discrete subgroup of the additive group $\mathbb{R}^n$ which spans the real vector space $\mathbb{R}^n$.

# Fragility of Pricing in Absence of Structural Stability

- We show that if structural stability does not hold, continuous superhedging pricing can be "fragile": it means that uniformly small (with respect to $d_{PH}$ metric) perturbations of the model's dynamics can dramatically change superhedging prices, i.e. approximation result like (4) fails.
- **The fragility of pricing is evident when a "no arbitrage" condition is broken, but it can be the case even if the "no arbitrage" condition is preserved by perturbations.**
- Below we formulate a result (not yet published) concerning the "fragility" of superhedging prices for one-step (i.e. $N = 1$) deterministic price dynamics model, when the condition NDAO holds, but RNDAO is not satisfied.

# Fragility Theorem

## Theorem 4

Assume that that there are no trading constraints, the number of risky assets $n \geq 2$, the initial deterministic model is defined by a continuous multifunction $K_1(\cdot)$ with (non-void) compact convex values such that NDAO holds and the set[a] $B_0^* = \{x \in B_0 : \ n - 1 \geq \dim(\mathrm{span}(K_1(\cdot))) > 0\}$ is non-void[b]. Then there exist a continuous payoff function $g_1$, a number $\beta > 0$ and a non-void Borel subset $B_0^{**}$ of $B_0^*$, such that for arbitrarily small $\delta > 0$ there exist a perturbed model satisfying NDAO, defined by a Borel-measurable multifunction $\tilde{K}_1(\cdot) = \tilde{K}_1^{(\delta)}(\cdot)$ with (non-void) compact convex values from, uniformly close to initial model in the Pompeiu–Hausdorff metric: $\sup\limits_x d_{PH}(\tilde{K}_1(x), K_1(x)) < \delta$, whereas the solutions of (1), $v_t^*$ and $\tilde{v}_t^*$ for initial and perturbed models respectively, differs: $v_t^*(x) - \tilde{v}_t^*(x) \geq \beta$ for all $x \in B_0^{**}$.

[a] Here $\mathrm{span}(A)$ stands for linear span (linear hull) of a set $A \subseteq \mathbb{R}^n$.

[b] Hence, RNDAO is not satisfied.

# “Realistic” Market Model and Structural Stability

# "Realistic" Stochastic Market Model

We regard as "realistic" a stochastic model of market behaviour if the conditional distributions of the current price depend "continuously" on the price history. This is a natural property of market models from an economic point of view: **there are no economic grounds for discontinuity of the dependence on price prehistory**. The following is a formalisation of this property.

## Definition

We say that a stochastic model of price evolution is realistic, if the transition kernels $Q_t$ corresponding to the conditional probabilities of the price $X_t \in \mathbb{R}^n$ at time $t$ with a known history $\bar{X}_{t-1} = \bar{x}_{t-1} \in (\mathbb{R}^n)^t$ admit a version satisfying the Feller property, i.e. the mapping $x \mapsto P_t(x, \cdot)$ is continuous in the weak topology on the space of probability measures.

# "Realistic" Deterministic Market Model

In the context of the deterministic approach, we have proposed the following formalisation of a "realistic" model of price evolution. Consider a financial market model with deterministic evolution of prices, where possible price increments at time $t$ are described by[12] $K_t(x)$, given price prehistory $x$; or, equivalently, possible price values at time $t$ are described by $F_t(x) = x + K_t(x)$.

## Definition

We call this model realistic if there exist mixed market strategies $P_t(x, \cdot)$ representable as Feller transition kernels,[a] satisfying the consistency condition, which is a relation between stochastic and deterministic models:

$$\mathrm{supp}(P_t(x, \cdot)) = F_t(x), \quad x \in B_{t-1} \text{ for } t = 1, \ldots, N.$$

[a] This can be interpreted as a smooth version of conditional distributions $X_t$ given prehistory $\bar{X}_{t-1} = x$.

[12] In this case, $K_t(\cdot)$ are not necessarily compact (but closed).

# Criterion of Realism for Deterministic Market Model

The class of realistic deterministic models is completely characterised by a result from[13]

- S. N. Smirnov, A Feller transition kernel with measure supports given by a set-valued mapping, *Proceedings of the Steklov Institute of Mathematics* **308** (2020) S188–S195.

## Theorem 5

The necessary and sufficient condition for the existence of such a (Feller) selector $P_t$ (that is described in the previous definition) is lower semicontinuity of the multifunctions[a] $F_t$.

[a] Or, equivalently, lower semicontinuity of the multifunctions $K_t$.

[13] In fact, in this paper we consider more general topological spaces and the sufficient condition is stronger then the necessary one; however, they coincide in the case of $\mathbb{R}^n$.

# Structural Stability and Kernels Property: Strong Feller in the Strict Sense

In what follows, we assume that there are no trading constraints, i.e. $D_t(\cdot) \equiv \mathbb{R}^n$. It turns out that under the assumption of structural stability, i.e. the RNDAO condition, and some stronger assumptions about the multifunctions $K_t(\cdot)$, the assertion of Theorem 5 can be strengthened[14].

## Theorem 6

If the set of initial prices $K_0$ is convex and compact, the values of the continuous multifunctions $K_t(\cdot)$ are convex compact sets, and RNDAO holds, then there exists a stochastic model, satisfying the consistency condition, such that the kernels $P_t$ are strong Feller in the strict sense.[a]

[a] A kernel $P$ is strong Feller in the strict sense iff the mapping $x \to P(x, \cdot)$ is continuous in the metric on the space of probability measures generated by the total variation norm

[14] Moreover, it can be shown in a constructive manner; note that the module of continuity for constructed Feller kernel (considering dependence of the probabilities on price prehistory) depends on the guaranteed (worst) structural stability threshold.

# Preservation of Structural Stability for Stochastic Models

# Preservation of Structural Stability in Terms of Probability Metrics

As the 2021 session of the International Seminar on Stability Problems for Stochastic Models commemorates the 90th birthday of the outstanding mathematician Vladimir Zolotarev (27.02.193–07.11.2019), founder of this seminar, we would like to take the opportunity to mention Zolotarev's considerable contribution to the theory of probability metrics, see e. g.

- V. M. Zolotarev, Probability metrics, *Theory of Probability and its Applications* **28**:2 (1983) 264–287.

We show the **preservation of structural stability for transitional kernel perturbations, small enough with respect to one of the three probability metrics considered** in

- S. N. Smirnov, Realistic models of financial market and structural stability, *Journal of Mathematics* [to appear]

This is one more argument in favour of the importance of structural stability for financial modelling.

# Notations

For a monotone nondecreasing function $f : [0, \infty) \to \mathbb{R}$, we define the upper generalised inverse by

$$f^{\lfloor -1 \rfloor}(y) = \inf\{x \in [0, \infty) : f(x) > y\} = \sup\{x \in [0, \infty) : f(x) \leq y\},$$

which is nondecreasing right-continuous.

Denote the open half-space with normal vector $u$ by $H_u = \{y \in \mathbb{R}^n : \langle u, y \rangle > 0\}$, the $\epsilon$-neighbourhood of the set $B$ by $B^\epsilon = \{z \in \mathbb{R}^n : \rho(z, B) < \epsilon\}$, where $\rho$ is Euclidian metric on $\mathbb{R}^n$ and $\rho(z, B) = \inf_{x \in B} \rho(z, x)$, $\mathcal{B}^n$ is Borel $\sigma$-algebra and $\mathcal{C}_{\mathbb{R}^n}$ is the class of all non-void convex subsets of $\mathbb{R}^n$.

# Probability Metrics Considered

Consider three metrics[15] on the space of probability measures on $(\mathbb{R}^n, \mathcal{B}^n)$,

$$d_{UC}(Q', Q) = \sup_{A \in \mathcal{C}_{\mathbb{R}^n}} |Q'(A) - Q(A)|,$$

$$l_\infty(Q', Q) = \inf\{\epsilon > 0 : Q(B) \le Q'(B^\epsilon),\ Q'(B) \le Q(B^\epsilon) \text{ for all } B \in \mathcal{B}^n\};$$

$$d_P(Q', Q) =$$
$$\inf\{\epsilon > 0 : Q(B) \le Q'(B^\epsilon) + \epsilon,\ Q'(B) \le Q(B^\epsilon) + \epsilon \text{ for all } B \in \mathcal{B}^n\}.$$

[15] Note that $l_\infty$ is the minimal metric with respect to the metric on the space of random vectors defined by $L_\infty(X', X) = \text{vrai}\max \rho(X', X)$, The metric $d_{UC}$ is a generalisation of the Kolmogorov metric $d_K$ defined for probabilities on the real line $\mathbb{R}$; on $\mathbb{R}$, these two metrics are equivalent: $d_K \le d_{UC} \le 2d_K$.The metric $d_{UC}$ is used, e.g., to estimate the speed of convergence in the multidimensional central limit theorem. The Prokhorov metric $d_P$ metrises the weak topology on the space of probabilities on a separable metric space.

# Theorem on Preservation of Structural Stability in Terms of Probability Metrics

## Theorem 7

Let $Q_t$ be the kernels of the initial model and $\tilde{Q}_t$ those of the perturbed model. Assume that there are no trading constraints and that for the initial model RNDAO holds (using the consistency condition).

$1^\circ$. Then[a] the perturbed model with kernels $\tilde{Q}_t$ satisfies RNDAO if it is close to the initial one in the sense that

$$d_{UC}(\tilde{Q}_t(x,\cdot), Q_t(x,\cdot)) < p_t^*(x),$$

where

$$p_t^*(x) = \inf_{u \in S_1(0)} Q_t(x, H_u) > 0$$

.

[a] Here the supports $Q_t(x,\cdot)$ and $\tilde{Q}_t(x,\cdot)$ are not necessarily compact and the result generalise that of Ostrovski cited above.

# Theorem on Preservation of Structural Stability in Terms of Probability Metrics (Continuation)

## Theorem 7 (continuation)

$2^\circ$. Suppose that the supports of $\tilde{Q}_t(x,\cdot)$ and $Q_t(x,\cdot)$ are compact. Then the perturbed model with kernels $\tilde{Q}_t$ satisfies RNDAO if it is close to the initial one in the sense that[a]

$$l_\infty(\tilde{Q}_t(x,\cdot), Q_t(x,\cdot)) < r_t^*(x),$$

where

$$r_t^*(x) = r(K_t(x)) > 0$$

and $r$ is given by (2).

[a] The RNDAO condition (for the initial model) is tantamount to the positivity of the structural stability threshold, i. e. $r(K_t(x)) > 0$ for all $x$ and $t$.

# Theorem on Preservation of Structural Stability in Terms of Probability Metrics (Continuation 2)

## Theorem 7 (continuation 2)

3°. There exists $d_t^*(x) \in (0,1)$ that is defined, given the measure $Q_t(x,\cdot)$, such that the perturbed model with kernels $\tilde{Q}_t$ satisfies RNDAO if[a]

$$d_P(\tilde{Q}_t(x,\cdot), Q_t(x,\cdot)) < d_t^*(x).$$

If the supports of $Q_t(x,\cdot)$ are compact, then it is sufficient to set [b]

$$d_t^*(x) = \phi_{t,x}^{\lfloor -1 \rfloor}(r(K_t(x))),$$

$$\phi_{t,x}(z) = z + \psi_{t,x}^{\lfloor -1 \rfloor}(z),$$

$$\psi_{t,x}(u) = \inf_{y \in K_t(x)} Q(x, \bar{B}_u(y)).$$

[a] In this case the supports $Q_t(x,\cdot)$ and $\tilde{Q}_t(x,\cdot)$ are not necessarily compact.

[b] Recall that $r$ is given by (2).

Thank you for your attention