Continuous time random walks modeling of quantum measurement and fractional equations of quantum stochastic filtering and control

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The talk will be based mostly on the author's preprint [12].

1 Introduction

Direct continuous observations are known to destroy quantum evolutions (so-called quantum Zeno paradox), so that continuous quantum measurements have to be indirect, and the results of the observation are assessed via quantum filtering. Initially developed in the framework of quantum stochastic calculus by Belavkin in the 80s of the last century in [4], [5], the main equations of quantum stochastic filtering, often referred to as the Belavkin equations, were later on derived via more elementary approach, as the limit of standard discrete measurements under appropriate scaling, see e.g. [6], [17]. The scaling arises from the basic Markovian assumption that the times between measurement are either fixed or exponentially distributed, like in a standard random walk. Since such Markovian assumption has no a priori justification, in many branches of modern physics it became popular to extend random walk modeling to the continuous time random walk (CTRW) modeling, where the time between discrete events is taken to be non-exponential, usually from the domain of attraction of a stable law. In the present paper we apply the CTRW modeling to the continuous quantum measurements yielding the new fractional in time evolution equations of quantum filtering in the scaling limit. The related quantum control problems turn out to be described by the fractional Hamilton-Jacobi-Bellman (HJB) equations on Riemannian manifolds (complex projective spaces in the case of finite-dimensional quantum mechanics) or the fractional Isaacs equation in the case of competitive control. By-passing we provide a full derivation of the standard quantum filtering equations (explaining from scratch all underlying quantum mechanical rules used) in a slightly modified and simplified way yielding also new explicit rates of convergence (which are not available via the tightness of martingales approach developed previously) and tailored in a way that allows for the direct applications of the basic results of CTRWs to deduce the final fractional filtering equations.

Several general comments on a wider context are in order.

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(i) The fractional equations of quantum stochastic filtering derived here can be considered as an alternative formulation of fractional quantum mechanics, which is different from the framework of fractional Schrödinger equations suggested in [15] and extensively studied recently. This leads also to a different class of quantum control problems, as those related to fractional Schrödinger formulation.

(ii) The fractional versions of the classical stochastic filtering (see [1] for the basics) has been actively studied recently, see e.g. [20].

(iii) The quantum mean-field games as developed by the author in [11] can now be extended to the theory of fractional quantum mean-field games. The classical versions of fractional mean-field games just started to appear in the literature, see [7]. On the other hand, the application of classical stochastic filtering in the study of mean-field games has also started to appear, see [18].

(iv) Fractional modeling and CTRW become very popular in almost all domains of physics, as well as economics and finances, see e.g. [2], [19] for some representative references.

2 The starting point: Markov chains of sequential indirect observations

Here we describe the Markov chains of sequential indirect observations (rather standard by now, in discrete and continuous time recalling first quickly the main notions related to quantum measurements.

Physical observables are given by self-adjoint operators A in $\mathcal{H}$. If A has a discrete spectrum (which is always the case in finite-dimensional $\mathcal{H}$, that we shall mostly work with), then A has the spectral decomposition $A = \sum_j \lambda_j P_j$, where P_j are orthogonal projections on the eigenspaces of A corresponding to the eigenvalues λ_j. According to the *basic postulate of quantum measurement* , measuring observable A in a state γ (often referred to as the *Stern-Gerlach experiment*) can yield each of the eigenvalue λ_j with the probability

$$\mathrm{tr}\,(\gamma P_j) = \mathrm{tr}\,(P_j \gamma P_j), \tag{1}$$

and, if the value λ_j was obtained, the state of the system changes (instantaneously) to the reduced state

$$P_j \gamma P_j / \mathrm{tr}\,(\gamma P_j).$$

In particular, if the state ρ was pure, $\gamma = |\psi\rangle\langle\psi|$, then the probability to get λ_j as the result of the measurement becomes $(\psi_, P_j \psi)$ and the reduced state also remains pure and is given by the vector $P_j \psi$. If the interaction with the apparatus was preformed 'without reading the results', the state ρ is said to be subject to a *non-selective measurement* that changes γ to the state $\sum_j P_j \rho P_j$.

Indirect measurements of a chosen quantum system in the initial space $\mathcal{H}_0$, which we shall often referred to as an atom, are organised in the following way. One couples the atom with another quantum system, a measuring devise, specified by another Hilbert space $\mathcal{H}$. Namely the combined system lives in the tensor product Hilbert space $\mathcal{H}_0 \times \mathcal{H}$ and its evolution is given by certain self-adjoint operator H in $\mathcal{H}_0 \times \mathcal{H}$. In the measuring device some fixed vector $\varphi \in \mathcal{H}$ is chosen, called the vacuum and interpreted as the stationary state of the devise when no interaction is involved. The corresponding density matrix will be denoted $\Omega = |\varphi\rangle\langle\varphi|$. Indirect measurements of the states of the atom are performed by measuring the coupled system via an observable of the second system and then projecting the resulting state to the atom via the partial trace.

Namely it is described by an operator R in $\mathcal{H}$ with the spectral decomposition $R = \sum_j \lambda_j P_j$ and is performed in two steps: given a state γ in $\mathcal{H}_0 \times \mathcal{H}$ one performs a measurement of R lifted as $I \otimes R$ to $\mathcal{H}_0 \times \mathcal{H}$ yielding values λ_j and new states

$$(I \otimes P_j)\gamma(I \otimes P_j)/\mathrm{tr}\,(\gamma(I \otimes P_j))$$

with probabilities $p_j = \mathrm{tr}\,(\gamma(I \otimes P_j))$, and then one projects these states to $\mathcal{H}_0$ via the partial trace producing the states

$$\mathrm{tr}_{p1}[(I \otimes P_j)\gamma(I \otimes P_j)/\mathrm{tr}\,(\gamma(I \otimes P_j))]. \tag{2}$$

The discrete time *Markov chain of successive indirect observations* (or measurements) evolves according to the following procedure specified by a triple: a self-adjoint operator H in $\mathcal{H}_0 \times \mathcal{H}$, a self-adjoint operator R in $\mathcal{H}$ and the vacuum vector Ω in $\mathcal{H}$. (i) Starting with an initial state ρ of $\mathcal{H}_0$ one couples it with the device in its vacuum state Ω producing the state $\gamma = \rho \otimes \Omega$ in $\mathcal{H}_0 \times \mathcal{H}$, (ii) During a fixed period of time t one evolves the system according to the operator H producing the state $\gamma_t = e^{-itH}\gamma e^{itH}$ in $\mathcal{H}_0 \times \mathcal{H}$, (iii) One performs the indirect measurement with the state γ_t yielding the states

$$\rho_t^j = \mathrm{tr}_{p1}\frac{(I \otimes P_j)\gamma_t(I \otimes P_j)}{p_j(t)} = \mathrm{tr}_{p1}\frac{(I \otimes P_j)e^{-itH}(\rho \otimes \Omega)e^{itH}(I \otimes P_j)}{p_j(t)} \tag{3}$$

with the probabilities

$$p_j(t) = \mathrm{tr}\,(\gamma_t(I \otimes P_j)) = \mathrm{tr}\,(e^{-itH}(\rho \otimes \Omega)e^{itH}(I \otimes P_j)). \tag{4}$$

Then the same repeats starting with ρ_t as the initial state. Let us denote U_t the transition operator of this Markov chain that acts on the set of continuous functions on $S(H)$ as

$$U_t f(\rho) = \mathbf{E} f(\rho_t) = \sum_j p_j(t) f(\rho_t^j). \tag{5}$$

Similarly one can define the continuous time *Markov chain of successive indirect observations* (or measurements) $O_{t,\lambda}^{\rho}$ and the corresponding Markov semigroup T_t^{λ} on $C(H(S))$ evolving according to the same rules, with only difference that the times t between successive measurements are not fixed, but represent exponential random variables τ with some fixed intensity λ: $\mathbf{P}(\tau > t) = e^{-\lambda t}$. The generator L^{λ} of this Markov process is bounded in $C(S(H))$ and acts as

$$L^{\lambda} f(\rho) = \frac{(U_{\lambda} f - f)(\rho)}{\lambda} = \frac{1}{\lambda}\sum_j p_j(\lambda)(f(\rho_{\lambda}^j) - f(\rho)). \tag{6}$$

All "quantum content" of the theory is now captured in the explicit formula (3). What follows will be the pure classical probability analysis of these Markov chains, their scaling limits and control.

3 The results

We show that for the case of the so-called counting type observation, the generator of the limiting process of filtered states (quantum filtering process) is

$$L_{count}f(\rho) = -(f'(\rho), i[A,\rho] + \frac{1}{2}\{C^*C,\rho\} - \rho T) + T\left[f(\frac{C\rho C^*}{T}) - f(\rho)\right]. \quad (7)$$

We show that for the case of the diffusive (homodyne) type observation, the generator of the limiting process limiting process of filtered states (quantum filtering process) is

$$L_{dif}f(\rho) = \frac{1}{2}[(\rho C^* + C\rho - \Omega\rho)f''(\rho)(\rho C^* + C\rho - \Omega\rho)] + (f'(\rho), -i[A,\rho] - \frac{1}{2}\{C^*C,\rho\} + C\rho C^*). \quad (8)$$

The rates of convergence from the discrete measurement scheme are obtained.

If observation goes through different channel the generator becomes L_{mix}, which is the sum of terms related to counting and diffusion observation.

If times between discrete quantum measurements are not exponential and are modeled as CTRW, the limiting process becomes governed by the fractional equation

$$D^{(\nu)}_{0+\star}f_t(x) = L_{mix}f_t(x), \quad f_0(x) = f(x), \quad (9)$$

with D^ν is the generalized fractional derivative (for instance, the standard Caputo derivative of an order β).

4 Fractional quantum control and games

From the theory above one can naturally initiate the theory of quantum fractional dynamic control and games.

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