# On the Estimating the Rate of Convergence for $M_t^X/M_t^X/1$ Queueing Models

***A. I. Zeifman*[1], *Y. A. Satin*[2] *A. S. Sipin*[2]**

[1]Vologda State University; Federal Research Center Computer Science and Control of RAS; Vologda Research Center of RAS, Moscow Center for Fundamental and Applied Mathematics, Moscow State University, Russia, a_zeifman@mail.ru
[2]Vologda State University, Vologda, Russia, yacovi@mail.ru, cac1909@mail.ru

We consider, as in Zeifman et al [4], four classes of contiuous-time Markov chains $X(t)$ with the following transition intensities:

(i) $q_{ij}(t) = 0$ for any $t \geq 0$ if $|i-j| > 1$ and both $q_{i,i+1}(t) = \lambda_i(t)$ and $q_{i,i-1}(t) = \mu_i(t)$ may depend on $i$;

(ii) $q_{i,i-k}(t) = 0$ for $k > 1$, $q_{i,i-1}(t) = \mu_i(t)$ may depend on $i$; and $q_{i,i+k}(t)$, $k \geq 1$, depend only on $k$ and does not depend on $i$;

(iii) $q_{i,i+k}(t) = 0$ for $k > 1$, $q_{i,i+1}(t) = \lambda_i(t)$ may depend on $i$; and $q_{i,i-k}(t)$, $k \geq 1$, depend only on $k$ and does not depend on $i$;

(iv) both $q_{i,i-k}(t)$ and $q_{i,i+k}(t)$, $k \geq 1$, depend only on $k$ and do not depend on $i$.

Each such process can be considered as the queue-length process for the corresponding queueing system $M_t^X/M_t^X/1$.

Then type (i) transitions describe Markovian queues with possibly state-dependent arrival and service intensities (for example, the classic $M_n(t)/M_n(t)/1$ queue); type (ii) transitions allow consideration of Markovian queues with state-independent batch arrivals and state-dependent service intensity; type (iii) transitions lead to Markovian queues with possible state-dependent arrival intensity and state-independent batch service; type (iv) transitions describe Markovian queues with state-independent batch arrivals and batch service. We can refer to them as $M_t^X/M_t^X/1$ queueing model following the original paper Nelson [2] , see also Li, Zhang [1], Satin et al [3], Zeifman et al [4].

In this lecture we consider the problem of finding the upper bounds for the rate of convergence for weakly ergodic situation.

To obtain these estimates, we investigate the corresponding forward system of Kolmogorov differential equations. To do this, it must first be transformed in a special way, as is done in our previouse papers, see for instance Zeifman et al [4].

After such a transformation, a linear (finite or countable) system of the form $\frac{d\mathbf{x}}{dt} = B^*(t)\mathbf{x}$ is obtained.

The simplest and most convenient for studying the rate of convergence to the limiting regime is the method of the logarithmic norm, see Zeifman et al [4]. This method gives good results if the matrix $B^*(t)$ is essentially non-negative.

This is certainly the case for circuits of the first and second classes, and for the third and fourth classes it is only true if additional conditions are met.

It turns out that in such cases it is possible to use a different approach, namely the method of differential inequalities.

In the case of finite Markov chains with analitical (in $t$) intensity functions this approach has been described in Zeifman et al [4].

Here we consider the general case of a countable Markov chain, assuming that the transition intensities are locally integrable.

The corresponding estimates of the rate of convergence are obtained by approximating via smoother coefficients and truncations using finite Markov chains, see Zeifman et al [5] for a detailed discussion.

**Acknowledgement.** This research was supported by Russian Science Foundation under grant 19-11-00020.

### References

1. J. Li, L. Zhang. $M^X/M/c$ Queue with catastrophes and state-dependent control at idle time, *Frontiers of Mathematics in China* **12** (2017) 1427–1439.

2. R. Nelson, D. Towsley, A. Tantawi. Performance Analysis of Parallel Processing Systems, *IEEE Transactions on software engineering* **14**: 4 (1988) 532–540.

3. Y. Satin, A. Zeifman, A. Korotysheva. On the Rate of Convergence and Truncations for a Class of Markovian Queueing Systems, *Theory of Probability & Its Applications* **57** (2013) 529–539.

4. A. Zeifman, Y. Satin, A. Kryukova, R. Razumchik, K. Kiseleva, G. Shilova. On the Three Methods for Bounding the Rate of Convergence for some Continuous-time Markov Chains, *International Journal of Applied Mathematics and Computer Science* **30** (2020) 251–266.

5. A. Zeifman, Y. Satin, A. Sipin. Bounds on the Rate of Convergence for $M_t^X/M_t^X/1$ Queueing Models, https://arxiv.org/abs/2105.05703